Homogeneous 1st order ODE

Aside: Def Homogeneous Equation: A function $f(x)$ is said to be "homogeneous" of degree n if replacing $x \to \lambda x$ we find that

$$f(\lambda x) = \lambda^n f(x)$$

ex) In 2D $f(x,y) = 2x^2 - 3y^2 + 4xy$

Take $\begin{cases} x \to \lambda x \\ y \to \lambda y \end{cases}$

$$f(\lambda x, \lambda y) = 2(\lambda x)^2 - 3(\lambda y)^2 + 4\lambda x \lambda y$$

$$= \lambda^2 2x^2 - \lambda^2 3y^2 + \lambda^2 4xy$$

$$= \lambda^2 (2x^2 - 3y^2 + 4xy) = \lambda^2 f(x,y)$$

An ODE of 1st Order is said to be Homogeneous if the following is true:

$$(\mathcal{L}) \quad M(x,y)dx + N(x,y)dy = 0$$

if $M(\lambda x, \lambda y) = \lambda^n M(x,y)$

$N(\lambda x, \lambda y) = \lambda^n N(x,y)$

To solve $(\mathcal{L})$ if homogeneous:

let $y = ux$ (solve for $u(x)$)

or $x = vy$ (solve for $v(y)$)

then "go back" to $y(x)$.

Will yield a separable ODE 1st order.

ex) $\underbrace{2x^3y}_{M(x,y)}\,dx + \underbrace{(x^4+y^4)}_{N(x,y)}\,dy = 0$ (ODE)

Is it homogeneous?

let $x \to x\lambda$ $y \to \lambda y$

and substitute into M & N

$$M(\lambda x, \lambda y) = 2(\lambda x)^3(\lambda y) = \lambda^4 2x^3 y = \lambda^4 M(x,y)$$

$$N(\lambda x, \lambda y) = \lambda^4 x^4 + \lambda^4 y^4 = \lambda^4 (x^4 + y^4) = \lambda^4 N(x,y)$$

let $y = u(x)x$ ($\cancel{**}$)

(want to get eq in u, x)

($\cancel{*}$) $dy = du\, x + u\, dx$

Before making the replacement:

$$2x^3 y\, dx + x^4\left(1 + \left(\tfrac{y}{x}\right)^4\right) dy = 0$$

Divide by x^3:

(*) $2y\, dx + x(1+u^4)\, dy = 0$ subst $\left\{\begin{array}{l} y = ux \\ \cancel{*} \end{array}\right.$ into (*)

$2ux\, dx + x(1+u^4)(du\, x + u\, dx) = 0$ divide by x

$2u\, dx + (1+u^4)(du\, x + u\, dx) = 0$ expend:

$2u\, dx + x(1+u^4)\, du + u(1+u^4)\, dx = 0$ group:

$\underbrace{[2u + u(1+u^4)]}_{3u+u^5}\, dx = -x(1+u^4)\, du$ divide by x

divide by $3u+u^5$

$$\frac{dx}{x} = -\frac{(1+u^4)\, du}{3u+u^5}$$

Separable so integrate b.s.

(∃) $\ln|x| = -\int \frac{(1+u^4)\,du}{3u+u^5} + c = -\frac{1}{6}\ln(u^4+3) - \frac{1}{3}\ln|u| + c$

The integral is not easy to do:

$\int \frac{(1+u^4)\,du}{u(3+u^4)}$ let $w = 3+u^4 \therefore u^4 = w-3$

$dw = 4u^3\,du \Rightarrow du = \frac{dw}{4u^3}$

$$\int \frac{(1+w-3)}{uw}\frac{dw}{4u^3} = \frac{1}{4}\int\frac{(w-2)\,dw}{wu^4} = \frac{1}{4}\int\frac{(w-2)\,dw}{(w-3)w}$$

Use partial fractions (see)

$$\frac{1}{w(w-3)} = \frac{A}{w} + \frac{B}{w-3} \Rightarrow 1 = A(w-3) + Bw$$

$$\therefore 1 = (A+B)w - 3A \Rightarrow A = -\frac{1}{3} \quad B = -A = \frac{1}{3}$$

$$\therefore \frac{1}{4}\int\frac{w-2}{(w-3)w}dw = \frac{1}{12}\int\frac{2\,dw}{w} + \frac{1}{12}\int\frac{dw}{w-3}$$ both are logarithmic integrals

$\frac{2}{12}\ln|w| + \frac{1}{12}\ln|w-3| + c$ now replace $w = 3+u^4$

$$\frac{1}{6}\ln|3+u^4| + \frac{1}{12}\ln|u^4| + c = \frac{1}{6}\ln|3+u^4| + \frac{1}{3}\ln|u| + c$$

$\therefore$ Going back to (∃):

Need to replace $u = y/x$

$$\ln x = -\frac{1}{6}\ln\left(\left(\frac{y}{x}\right)^4 + 3\right) - \frac{1}{3}\ln\left(\frac{y}{x}\right) + C$$

$$\ln x + \frac{1}{6}\ln\left(\frac{3x^4 + y^4}{x^4}\right) + \frac{1}{3}\ln\left(\frac{y}{x}\right) = C$$

$$\ln x + \ln\left[\left(\frac{3x^4 + y^4}{x^4}\right)\right]^{1/6} + \ln\left[\left(\frac{y}{x}\right)^{1/3}\right] = C$$

Not sure if we want to go any further with // simplification...

As a check, see if the substitution $x = vy$ leads to $\frac{2v^3\,dv}{1+3v^4} = -\frac{1}{y}dy$, which are both easy to integrate: $\frac{1}{6}\int \frac{dw}{w} = -\ln|y| + c$ or $\frac{1}{6}\ln|w| + \ln|y| = C$

$\begin{cases} w = 1+3v^4 \\ dw = 12v^3\,dv \end{cases}$ or $\frac{1}{6}\ln|1+3v^4| + \ln|y| = c$

or

Hence $\frac{1}{6}\ln\left(1+3\left(x/y\right)^4\right) + \ln y = C$

Suppose $(\dagger)\ \frac{dy}{dx} = f(x,y) = g\left(y/x\right)$

then $u = y/x \Rightarrow y = ux$

substitute in $(\dagger)$

$$\frac{dy}{dx} = g(u) \qquad A$$

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$ substitute into A:

$$\frac{du}{dx}x + u = g(u)$$

$$x\frac{du}{dx} = g(u) - u$$

$$\frac{du}{g(u)-u} = \frac{dx}{x}$$

separable. Integrate for $u = u(x)$

then to find $y = u(x)\,x$. //

ex) $$\frac{dy}{dx} = \frac{x-y}{x+y} = f(x,y) \overset{?}{=} g(y/x)$$

$$\frac{dy}{dx} = \frac{x-y}{x+y}\frac{1/x}{1/x} = \frac{1-y/x}{1+y/x}$$

let $u = y/x \Rightarrow y = ux$

$$\frac{dy}{dx} = \frac{1-u}{1+u}$$

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

$$\frac{du}{dx}x + u = \frac{1-u}{1+u}$$

$$x\frac{du}{dx} = \frac{1-u}{1+u} - u = \frac{1-u-u(1+u)}{1+u}$$

$$x\frac{du}{dx} = \frac{1-2u-u^2}{1+u}$$

$$\frac{(1+u)\,du}{(1-2u-u^2)} = \frac{dx}{x}$$