AN EXAMPLE (SEPARABLE 1st ORDER ODE)

$$\frac{dN}{dt} = -N + Nte^{t+2}, \text{ where } N = N(t).$$

$$\frac{dN}{dt} = N(-1 + te^{t+2})$$

$$\frac{dN}{N} = -(1 - te^{t+2})\,dt.$$

Integrate both sides:

$$\ln|N| = -\int dt + \int te^{t+2}\,dt + C$$

$$\ln|N| = -t + e^{2}\underbrace{\int te^{t}\,dt}\, + C$$

Integrate by parts

$$\int te^{t}\,dt = te^{t} - \int e^{t}\,dt = te^{t} - e^{t} + C,$$

$u = t$ $v = e^{t}$

$du = dt$ $dv = e^{t}\,dt$

IBP: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

$$\ln|N| = -t + e^2\left[te^t - e^t\right] + C$$

exponentiating,

$$|N| = Ke^{-t+e^2[te^t - e^t]}$$, K is a constant, or

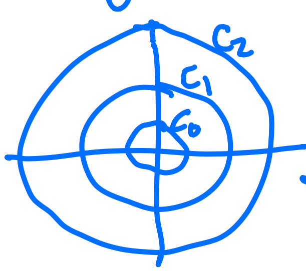
$$N = Ke^{-t+e^{2+t}[t-1]}.$$

The above is an explicit expression. Consider the following separable ODE:

$$y\,dy = -x\,dx$$ integrating b.s.

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + C.$$ Hence,

☆ $x^2 + y^2 = C$ "General Solution" is an implicit relation (because it's a graph)



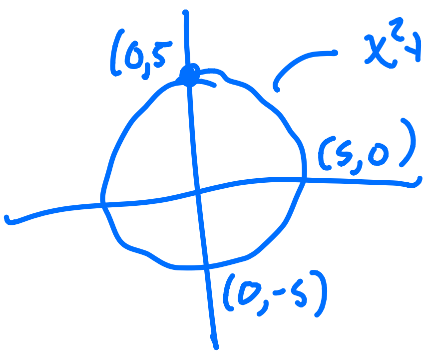
the c_i "chooses the circle":

Suppose $y(0) = 5$ I.C. (or constraint)

The I.C. will pick out a member of the (*) family:

When $x=0$ $y=5$: $y(0)=5$

$$0^2+5^2=C \Rightarrow C=25$$



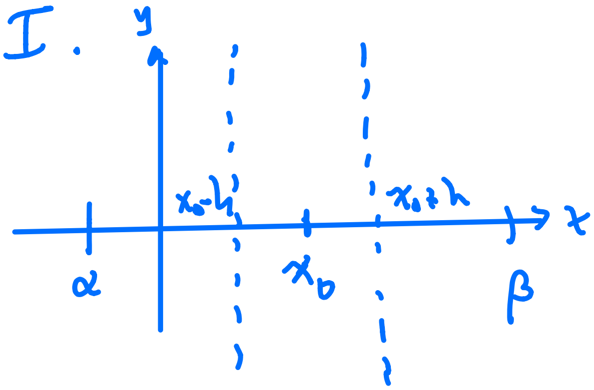
$x^2+y^2=25$ a particular solution

How do we know that a 1st order ODE has a solution? How do we know if there is one or many?

<u>Thm</u>: Suppose $\frac{dy}{dx}=f(x,y)$. Let $y=y(x)$.

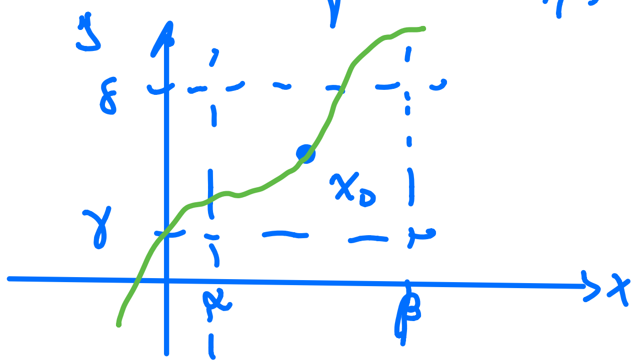
Let f be continuous on a rectangle $\alpha<x<\beta$, $\gamma<y<\delta$. Also assume that $y(x_0)=y_0$ is also in $\gamma<y<\delta$ and $\alpha<x_0<\beta$.

Then, for some interval $I = (x_0 - h, x_0 + h)$ $\in (\alpha, \beta)$ there exists a solution $y = y(x)$ in I.



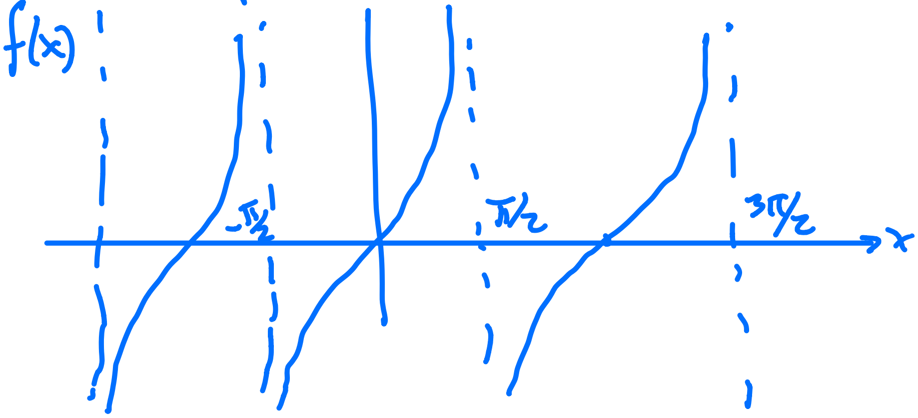
Th'm Uniqueness: As above. Further, assume that $\dfrac{\partial f}{\partial y}$ is continuous on $(\alpha, \beta) \times (\gamma, \delta)$

then solution is unique in $(\alpha, \beta) \times (\gamma, \delta)$



ex) $\dfrac{dy}{dx} = \tan x = f(x)$

Determine a range that guarantees finding a solution. Is the solution unique as well?



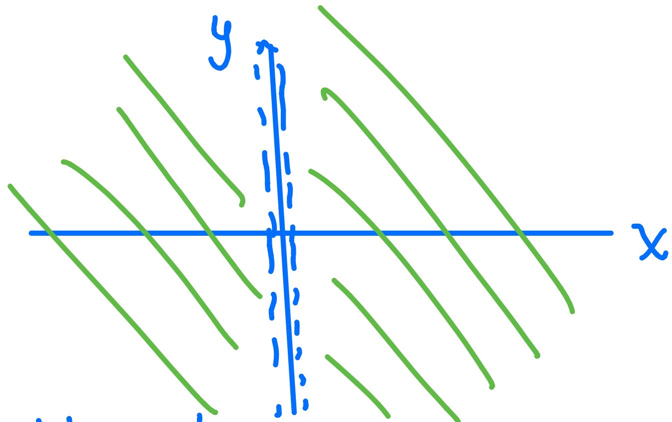
We are guaranteed a solution for $\pi/2 < x < \frac{3\pi}{2}$. Could also choose $-\pi/2 < x < \frac{\pi}{2}$ etc.

$\frac{\partial}{\partial y} \tan x = 0$ so continuous $\forall y$

so a solution is unique for $-\pi/2 < x < \pi/2$ and all y.

ex) $\begin{cases} \frac{dy}{dx} = \frac{4y}{x} = f(x,y) \\ y(x_0) = y_0 \end{cases}$

$\frac{\partial f}{\partial y} = \frac{4}{x}$



for $x \neq 0$.

Solving,

$$\frac{dy}{y} = \frac{4}{x}\,dx \quad \text{integrate b.s.}$$

$$\ln|y| = 4\ln|x| + c$$

$$\ln|y| = \ln|x|^4 + c$$

$$y = Kx^4 \quad (\ddagger)$$

Let's bring in the I.C.

$$y_0 = K x_0^4 \Rightarrow K = y_0 / x_0^4$$

K is constrained by requiring that $(\ddagger)$ pass through the point (x_0, y_0).