ex) $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$ think of this as the product of 2 functions in s-space

$$\mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s}\right) = 1 * 1 = \int_0^t d\tau = t$$

//

ex) $$\mathcal{L}^{-1}\left(\frac{2}{(s-1)(s+2)}\right) = \mathcal{L}^{-1}\left(\frac{2}{s-1}\frac{1}{s+2}\right)$$

$$= 2\, e^t * e^{-2t} = 2\int_0^t e^{-2\tau} e^{(t-\tau)} d\tau$$

$$= 2e^t \int_0^t e^{-2\tau} e^{-\tau} d\tau = 2e^t \int_0^t e^{-3\tau} d\tau$$

$$= 2e^t \left(-\frac{1}{3}\right) e^{-3\tau}\Big|_0^t = \frac{2}{3}\left(e^t - e^{-2t}\right)$$

//

ex) $y'' + y = g(t) \Rightarrow Ly = g \Rightarrow y = L^{-1}g$

$y(0) = y'(0) = 0$

Use $\mathcal{L}$ to solve. Let

$\mathcal{L}(y(t)) = Y(s)$ $\quad \mathcal{L}(g(t)) = G(s)$. Then

$$s^2 Y(s) + Y(s) = G(s)$$

$$Y(s)(s^2+1) = G(s)$$

$$Y(s) = \frac{G(s)}{s^2+1} = F(s)\,G(s) \text{ where}$$

$$F(s) = \frac{1}{s^2+1} \qquad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$y(t) = \int_0^t g(\tau) f(t-\tau)\,d\tau$$

$$y(t) = \int_0^t g(\tau) \sin(t-\tau)\,d\tau$$

$$y(t) = g * \sin t$$

//

INTEGRAL EQUATIONS

"The Volterra Equation"

Find a function $\varphi(t)$ such that

$$\varphi(t) + \int_0^t (t-\tau)\,\varphi(\tau)\,d\tau = 1$$

To solve, use Laplace transforms:

$$\mathcal{L}\left[\varphi + \int_0^t (t-\tau)\,\varphi(\tau)\,d\tau\right] = \mathcal{L}(1)$$

$$\mathcal{L}(\varphi) = \Phi(s)$$

$$\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{1}{s}$$

$$\Phi(s)\left[1 + \frac{1}{s^2}\right] = \frac{1}{s}$$

$$\Phi(s)\left[\frac{s^2+1}{s^2}\right] = \frac{1}{s}$$

$$\Phi(s) = \frac{1}{s}\,\frac{s^2}{s^2+1} = \frac{s}{s^2+1}$$

From table $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$

$$\varphi(t) = \cos t$$

EXAMPLE OF INTEGRO-DIFFERENTIAL EQUATION

ex)
$$\begin{cases} \dfrac{d\varphi}{dt}(t) + 2\varphi(t) + \displaystyle\int_0^t \varphi(\tau)\,d\tau = 1 \quad (*) \\ \varphi(0) = 0 \end{cases}$$

$$\Phi(s) = \mathcal{L}(\varphi(t))$$

Take Laplace Transform of $(*)$

$$s\Phi(s) - \underbrace{\varphi(0)}_{=0} + 2\Phi(s) + \mathcal{L}\left[\int_0^t 1\,\varphi(\tau)\,d\tau\right] = \frac{1}{s}$$

$$s\Phi(s) + 2\Phi(s) + \frac{1}{s}\Phi(s) = \frac{1}{s}$$

$$\Phi(s)\left[s + 2 + \frac{1}{s}\right] = \frac{1}{s}$$

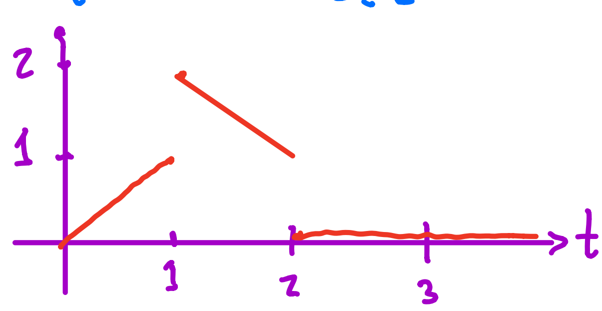
$$\Phi(s)\left[s^2 + 2s + 1\right] = 1$$

$$\Phi(s) = \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2} = e^{-t}t$$

In table

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ $\qquad$ $\mathcal{L}\left(e^{at}f(t)\right) = F(s-a)$ //

ex) $f(t) = \begin{cases} t & 0 \le t < 1 \\ 3-t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$



$$g(t) = t\left[\mathcal{U}(t) - \mathcal{U}(t-1)\right] + (3-t)\left[\mathcal{U}(t-1) - \mathcal{U}(t-2)\right]$$

From table $\mathcal{L}\left[\mathcal{U}(t-c)\, f(t-c)\right] = e^{-cs} F(s)$

and $\mathcal{L}\left[\mathcal{U}(t-c)\right] = \dfrac{e^{-cs}}{s}$

$$g(t) = t\,\mathcal{U}(t) +$$

$$\mathcal{U}(t-1)\left[-t + 3 - t\right] +$$

$$\mathcal{U}(t-2)\left[t-3\right]$$

$$G(s) = \frac{1}{s^2} - \mathcal{L}\left[\mathcal{U}(t-1)[2t-3]\right]$$

$$+ \mathcal{L}\left[\mathcal{U}(t-2)[t-2-1]\right]$$

$$= \frac{1}{s^2} + \mathcal{L}\left[\mathcal{U}(t-1)[2t-3]\right] + \mathcal{L}\left[\mathcal{U}(t-2)(t-2)\right]$$

$$- \mathcal{L}\left[\mathcal{U}(t-2)\right]$$

$$= \frac{1}{s^2} - \mathcal{L}\left[\mathcal{U}(t-1)[2t-3]\right] + e^{-2s}\frac{1}{s^2} - \frac{e^{-2s}}{s}$$

$$= \frac{1}{s^2} - \mathcal{L}\left[2\mathcal{U}(t-1)\left[t - \frac{3}{2}\right]\right] + \frac{e^{-2s}}{s^2} - \frac{1}{s}e^{-2s}$$

$$= \frac{1}{s^2} - 2\,\mathcal{L}\left[\mathcal{U}(t-1)\left(t - \frac{3}{2}\right)\right] + \frac{e^{-2s}}{s^2} - \frac{1}{s}e^{-2s}$$

$$= \frac{1}{s^2} - 2\,\mathcal{L}\left[\mathcal{U}(t-1)(t-1) - \frac{1}{2}\mathcal{U}(t-1)\right] + \frac{e^{-2s}}{s^2} - \frac{1}{s}e^{-2s}$$

$$= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{1}{s}e^{-2s}$$

$$= \frac{1}{s^2}\left[1 - e^{-s}\right] - \frac{1}{s}\left[e^{-s} + e^{-2s}\right]$$

$$= \frac{1}{s^2}\left(1 - e^{-s}\right) - \frac{1}{s}e^{-s}\left[1 + e^{-3s}\right]$$