ex) $$y'' + 2y' + 2y = \sum_{l=1}^{3} k\,\delta(t - l\pi)$$

$y(0) = 0 \quad y'(0) = 1$

$k = e^{-\pi}$

Using Laplace Transforms:

$$s^2 Y(s) - s\,\overset{y(0)}{0} - \overset{y'(0)}{1} + 2sY(s) - 2\cdot\overset{y(0)}{0}$$
$$+ 2Y(s) = k\left[\sum_{l=1}^{3} e^{-l\pi s}\right]$$

$$\Rightarrow \left[s^2 + 2s + 2\right]Y(s) - 1 = k\sum_{l=1}^{3} e^{-l\pi s}$$

$$Y(s) = \frac{1 + k\sum_{l=1}^{3} e^{-l\pi s}}{s^2 + 2s + 2}$$

$$= \frac{1}{s^2+2s+2} + \frac{e^{-\pi}\sum_{l=1}^{3} e^{-l\pi s}}{s^2+2s+2}$$

$$Y(s) = \frac{1}{(s+1)^2+1} + \frac{e^{-\pi}\sum_{\ell=1}^{3} e^{-\ell\pi s}}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}(Y(s)) = e^{-t}\sin t + e^{-\pi}\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s+1)^2+1} + \frac{e^{-2\pi s}}{(s+1)^2+1} + \frac{e^{-3\pi s}}{(s+1)^2+1}\right]$$

$$= e^{-t}\sin t + e^{-\pi}\left\{\mathcal{U}(t-\pi)e^{-(t-\pi)}\sin(t-\pi) + \mathcal{U}(t-2\pi)e^{-(t-2\pi)}\sin(t-2\pi) + \mathcal{U}(t-3\pi)e^{-(t-3\pi)}\sin(t-3\pi)\right\}$$

THE CONVOLUTION OPERATOR

Let $f(t)$ and $g(t)$ be piecewise continuous on $t \in [0,\infty)$, then the CONVOLUTION of $f(t)$ & $g(t)$ is

$$h(t) = f*g = \int_0^t g(\tau) f(t-\tau) d\tau$$

Convolution symbol

The Convolution operator is linear:

suppose $l(t)$ is a piecewise continuous on $t \in [0, \infty)$ then

$$f*(g+l) = f*g + f*l$$

$$f*cg = c\, f*g$$

The Convolution obeys commutativity and associativity

$$f*g = g*f$$

and $$f*(g*l) = (f*g)*l$$

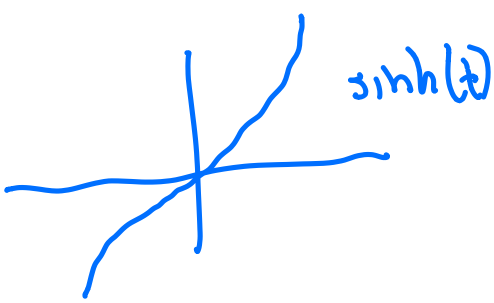
ex) $$f(t) * 1 = \int_0^t 1\, f(t-\tau)\,d\tau$$

$$= -\int_t^0 f(u)\,du = \int_0^t f(u)\,du$$

ex) $$e^t * e^{-t} = \int_0^t e^{-\tau} e^{t-\tau}\,d\tau$$

$$= e^t \int_0^t e^{-2\tau}\,d\tau = e^t \left(-\frac{1}{2} e^{-2\tau} \right|_0^t$$

$$= -\frac{1}{2} e^{-t} + \frac{1}{2} e^t = \sinh(t)$$



let confirm that

$f * g = g * f$ by direct computation:

$$f*g = \int_0^t g(\tau) f(t-\tau)\,d\tau$$

let $x = t-\tau \qquad \tau = t-x$

$dx = -d\tau$

$$= -\int_t^0 g(t-x) f(x)\,dx = \int_0^t g(t-\tau) f(\tau)\,d\tau$$

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THE CONVOLUTION THEOREM

Let $f(t)$ & $g(t)$ be piecewise continuous on $t \in [0, \infty)$. Let $\mathcal{L}$ be the Laplace transform:

$$\mathcal{L}\left(f(t) * g(t)\right) = F(s)\,G(s)$$

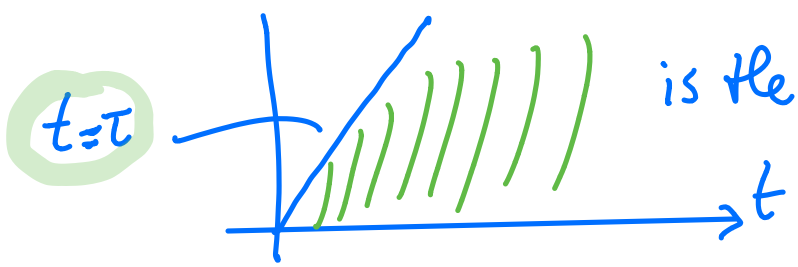
<u>Proof:</u>

$$\mathcal{L}\left\{ \int_0^t f(\tau)\, g(t-\tau)\, d\tau \right\}$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau \right] dt$$

Switch order of integration:

$$= \int_0^\infty f(\tau) \left[\int_0^\infty e^{-st} g(t-\tau)\, dt \right] d\tau$$



is the range of inside integral

$$\therefore \int_0^\infty f(\tau) \left[\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right] d\tau \qquad (\dagger)$$

So for the inner integral we get

Let $u = t - \tau \qquad t = u + \tau$

$du = dt$

$$\int_{\tau}^{\infty} e^{-st} g(t-\tau)\,dt = \int_{0}^{\infty} e^{-s(u+\tau)} g(u)\,du$$

$$= e^{-\tau s} \int_{0}^{\infty} e^{-su} g(u)\,du = e^{-\tau s} G(s)$$

Replace into $(\dagger)$

$$\int_{0}^{\infty} f(\tau) e^{-\tau s} G(s)\,d\tau = G(s) \int_{0}^{\infty} e^{-\tau s} f(\tau)\,d\tau$$

$$= G(s)\,F(s)$$

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