ex) IVP $\begin{cases} y'' + 4y = f(t) & t > 0 \\ y(0) = y'(0) = 0 \end{cases}$

$$f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ 0 & \pi \le t \end{cases}$$

$$f(t) = (1 - \mathcal{U}(t-\pi)) \sin t \quad \text{for } t \geq 0$$

$$= \sin t + \mathcal{U}(t-\pi) \sin(t-\pi)$$

$$F(s) = \mathcal{L}(f(t)) = \frac{1}{s^2+1} + e^{-\pi s} \frac{1}{s^2+1}$$

Transform (IVP)

Note: $\mathcal{L}(y'') = s^2 Y(s) - s y(0) - y'(0)$

$$s^2 Y(s) + 4Y(s) = F(s)$$

$$Y(s) = \frac{1}{s^2+4} F(s)$$

Aside: $Ly = f(t)$ $\quad L = \frac{d^2}{dt^2} + 4$

$y(t) = L^{-1} f(t) \iff$

$Y(s) = \mathcal{L}(L^{-1} f(t))$

Back to IVP:

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + e^{-\pi s} \frac{1}{(s^2+4)(s^2+1)}$$

Find $\mathcal{L}^{-1}(Y(s)) = y(t)$

By Partial Fractions

$$Y(s) = \frac{1}{3} \frac{1}{s^2+1} - \frac{1}{3} \frac{2/2}{s^2+4} + e^{-\pi s}\left[\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2/2}{s^2+4}\right]$$

Making use of Table:

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \mathcal{U}(t-\pi)\left[\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin(2(t-\pi))\right]$$

or

$$y(t) = \begin{cases} \frac{1}{3}\sin t - \frac{1}{6}\sin 2t & 0 \le t < \pi \\ -\frac{1}{3}\sin 3t & \pi \le t \end{cases}$$

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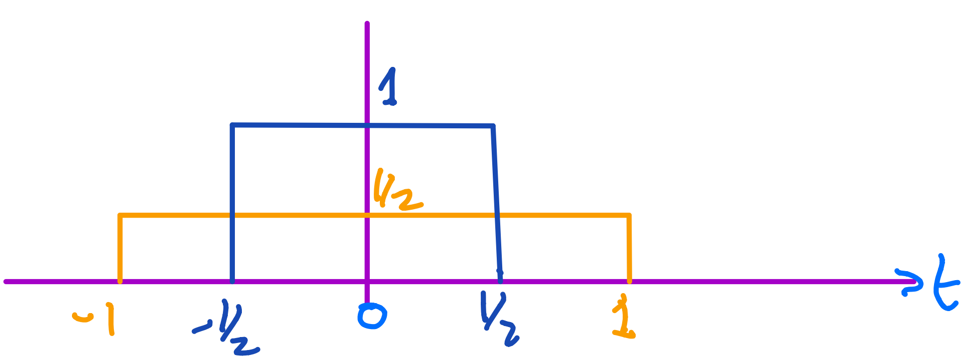
IMPULSE FUNCTIONS (DIRAC DELTA FUNCTION)

Sect 7.6

GENERALIZED FUNCTION $\delta_\tau(t)$:

We propose a function that depends on a small parameter, which when integrated will equal 1 (area = 1). Generalized function is not unique:

For example $$\delta_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$



τ is a parameter. It doesn't have to be small but we require $\tau > 0$. By construction:

$$\int_{-\infty}^{\infty} \delta_\tau(t)\,dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt = 1$$

We add another property: the generalized function must be continuous at $t=0$. However, there's nothing special about $t=0$. We can place δ_τ at $t=t_0$. To indicate this, we write:

$$\delta_\tau(t_0) \text{ or } \delta_\tau(t-t_0)$$

$\boxed{\text{The Sifting Property:} \quad \lim_{\tau\to 0} \int_{-\infty}^{\infty} \delta_\tau(t-t_0)\, f(t)\,dt = f(t_0) \quad \text{For } f(t) \text{ continuous around } t=t_0.}$

The Unit Impulse or Dirac Delta Function

It is called a "function" but this class of functions have special properties that make them amenable to the analysis of integrals:

The Dirac Delta function is called a DISTRIBUTION.

The connection between Dirac Delta & Generalized functions is in the limit as $\tau \to 0$ the generalized function inherits all the properties of the Dirac Delta.

DIRAC DELTA FUNCTION

$$\delta(t - t_0) = \begin{cases} \text{non-zero at } t = t_0 \\ 0 \quad \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

and obeys the sifting property:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$$

The Laplace Transform of Dirac Delta:

$$\mathcal{L}(\delta(t-t_0)) = \int_0^{\infty} \delta(t-t_0) e^{-st}\,dt = e^{-st_0}$$

$t_0 \geq 0$

It can be used to model physical phenomena:

ex) SHO, forced by an impulse external force:

$$m y'' + k y = p\,\delta(t-t_0)$$

θ has units of force

ODE $\quad y'' + \omega^2 y = \theta\,\delta(t-t_0) \quad \left\{ \begin{array}{l} t>0 \\ t_0>0 \end{array} \right.$

$\omega^2 = \frac{k}{m} \quad \theta = \frac{p}{m}$ is constant

Suppose that:

I.C. $\quad y(0) = y'(0) = 0$. We can solve ODE + IC:

Use Laplace Transforms:

$$s^2 Y(s) + \omega^2 Y(s) = \theta\,\mathcal{L}(\delta(t-t_0))$$

$$(s^2+\omega^2)\,Y(s) = \theta e^{-t_0 s}$$

$$Y(s) = \frac{\theta e^{-t_0 s}}{s^2+\omega^2}$$

Look it up in Table

$$y(t) = \mathcal{U}(t - t_0)\frac{\theta}{\omega}\sin\omega(t - t_0)$$

since $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$

and $\mathcal{L}(\mathcal{U}(t-t_0)f(t-t_0)) = e^{-st_0}F(s)$

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