Laplace Transform of a Derivative

$$\mathcal{L}(f'(t)) = \int_0^\infty \frac{df}{dt} e^{-st} ds$$

IBP $\quad u = e^{-st} \qquad v = f$

$\quad du = -se^{-st}dt \qquad dv = \frac{df}{dt}dt$

$$\mathcal{L}(f'(t)) = e^{-st}f\Big|_0^\infty + s\int_0^\infty f e^{-st}dt$$

$$= e^{-st}f\Big|_0^\infty + sF(s) = -f(0) + sF(s)$$

$$\mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0)$$

(can get also by IBP)

Generally:

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$$

ex) Find $\mathcal{L}(t\cos at) = \int_0^\infty t\cos at\, e^{-st}\,dt$

we could do this by IBP

However if $F(s) = \int_0^\infty f(t)e^{-st}\,dt$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty -t\,e^{-st} f(t)\,dt$$

$$\therefore -\frac{dF}{ds} = \mathcal{L}(t f(t))$$

Going back to $\mathcal{L}(t\cos at)$

From table $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} = F(s)$

$$\therefore \mathcal{L}(t\cos at) = -\frac{dF}{ds} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2} \; //$$

TRANSFORM OF PERIODIC FUNCTION

Assume $f(t)$ piecewise constant over $[0,\infty)$, periodic with period T

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

To verify:

$$\mathcal{L}(f(t)) = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

let $t = u+T \quad dt = du$

$$\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du$$

$$= e^{-sT} \int_0^\infty e^{-su} f(u)\,du = e^{-sT} F(s)$$

$$\mathcal{L}(f(t)) = F(s) = \int_0^T e^{-st} f(t)\,dt + e^{-sT} F(s)$$

$$F(s)\left(1 - e^{-sT}\right) = \int_0^T e^{-st} f(t)\,dt$$

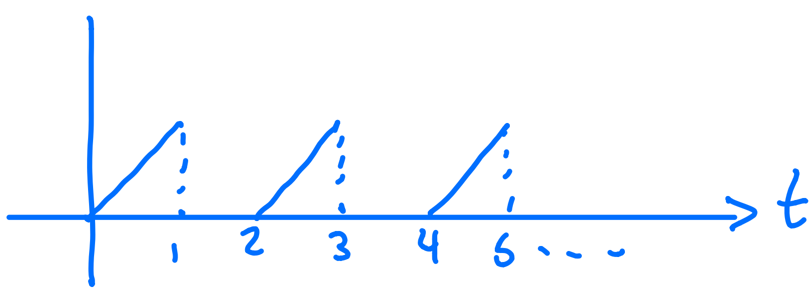
$$\therefore \quad F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt \quad //$$

ex) Find $\mathcal{L}(f(t))$ where

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

$g(t) = \text{periodic } f(t)$

$g(t) = f(t) \qquad g(t+T) = f(t)$



$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} t[\mathcal{U}(0) - \mathcal{U}(t-1)]\,dt$$

$$= \int_0^1 e^{-st} t\,dt \qquad \text{IBP}$$

$$w = t \qquad v = -\frac{1}{s}e^{-st}$$

$$dw = dt \qquad dv = e^{-st}dt$$

$$\int_0^1 e^{-st} t\,dt = -\frac{t}{s}e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt$$

$$= -\frac{t}{s}e^{-st}\Big|_0^1 - \frac{1}{s^2}e^{-st}\Big|_0^1$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}$$

$$\therefore \quad \mathcal{L}(g(t)) = \frac{1}{1-e^{-2s}}\left\{-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}\right\}$$

since $T = 2$

$$\mathcal{L}(g(t)) = \frac{1-(s+1)e^{-s}}{s^2(1-e^{-2s})} \quad //$$

LAPLACE TRANSFORMS USEFUL IN SOLVING LINEAR IVP

IVP = initial value problems

ex) $\begin{cases} y'' - y = e^t & \text{ODE} \quad y = y(t) \\ y(0) = 1 \quad y'(0) = 0 & \text{I.C.} \end{cases}$

$$\mathcal{L}(y'') = s^2 Y(s) - s\,y(0) - y'(0)$$

$$= s^2 Y(s) - s \quad \text{using I.C.}$$

$$\mathcal{L}(y'' - y) = \mathcal{L}(e^t) \quad \mathcal{L}(\text{ODE})$$

$$s^2 Y(s) - s - Y(s) = \frac{1}{s-1}$$

$$(s^2 - 1)\,Y(s) = s + \frac{1}{s-1}$$

$$Y(s) = \frac{s}{s^2-1} + \frac{1}{(s^2-1)(s-1)} = \mathrm{I} + \mathrm{II}$$

To find $y(t)$, use algebra (partial fractions) & Table.

$$\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) = \cosh t$$ this is $\mathcal{L}^{-1}(I)$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2-1)(s-1)}\right)$$ use Partial fractions

$$\frac{1}{(s^2-1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1 = A(s-1)^2 + B(s+1)(s-1) + C(s+1)$$

$$1 = A(s^2-2s+1) + B(s^2-1) + C(s+1)$$

$$s^2: \quad 0 = A+B \Rightarrow A = -B$$

$$s^1: \quad 0 = -2A + C \Rightarrow C = 2A = -2B$$

$$s^0: \quad 1 = A - B + C$$

$$1 = -B - B - 2B = -4B$$

$$B = -\frac{1}{4}$$

$$A = \frac{1}{4}$$

$$C = \frac{1}{2}$$

$\therefore$

$$\frac{1}{(s^2-1)(s-1)} = \frac{1/4}{s+1} + \frac{-1/4}{s-1} + \frac{1/2}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2-1)(s-1)}\right) = \frac{1}{4}e^{-t} - \frac{1}{4}e^{t} + \frac{1}{2}te^{t}$$

Combining the $\mathcal{L}^{-1}(\mathrm{I}) + \mathcal{L}^{-1}(\mathrm{II})$

$$\mathcal{L}^{-1}(Y(s)) = \frac{1}{4}e^{-t} - \frac{1}{4}e^{t} + \frac{1}{2}te^{t} + \cosh t$$

$$= -\frac{1}{2}\left(\frac{e^{t} - e^{-t}}{2}\right) + \cosh t + \frac{1}{2}te^{t}$$

$$y(t) = -\frac{1}{2}\sinh t + \cosh t + \frac{1}{2} t e^t$$