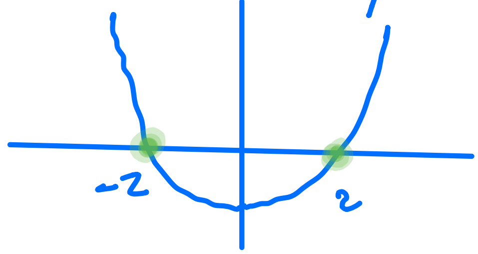
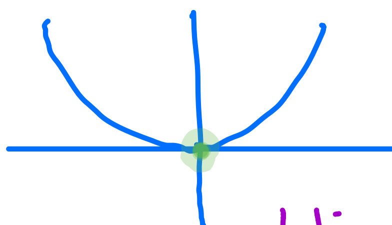
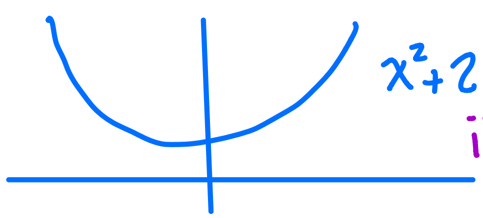
Nothing special about an equation having more than one solution, or not having a solution:

For example, $x^2-4=0$ has 2 solutions, $x=2$ and $x=-2$, they are called "roots".





$x^2=0$ has 1 solution. We say it is "unique" $x=0$ (it's really 2 solutions, but they are both 0)



$x^2+2=0$ has no real solutions. Actually, it has 2 solutions but in the complex plane.

For ODE's can obtain a family of solutions

$$\frac{dy}{dx}=f(x,y)$$ ODE, first order,

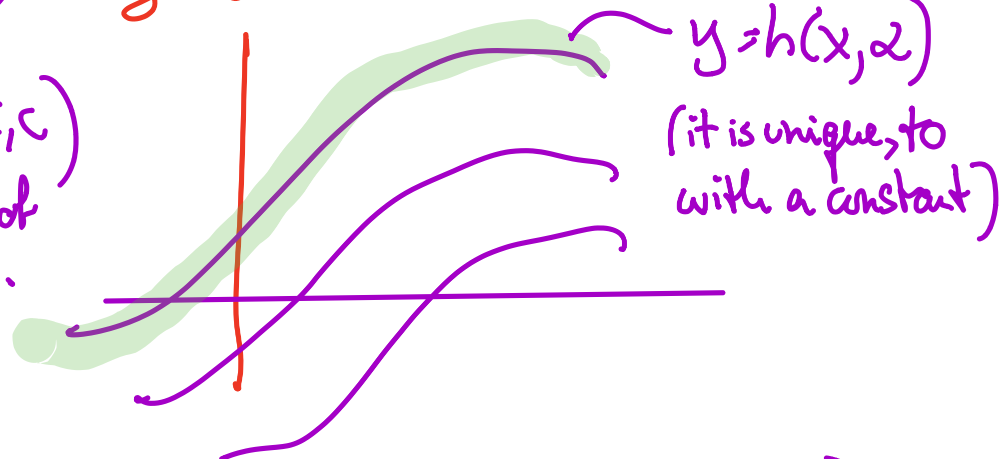
(★) $y=h(x,c)$, a function,

where c is undetermined constant $\Rightarrow$

(☆) a general solution & it is unique.

ex) $y = h(x, c)$

a family of solutions.



A particular solution is $y = h(x, c = \alpha)$

α is known constant. A member of the family.

An ODE may not have a solution, may have one (unique) solution, or many (non-unique).

N.B. For linear ODE's, if a solution is found, it will be unique (general or particular).

LINEAR vs. NonLINEAR ODE'S:

The ODE

$$f\left(x, y(x), y'(x) \ldots y^{(n)}(x)\right) = 0$$

is linear if the terms $y, y', y'' \ldots y^{(n)}$ appear to first power

ex) $y''' + ay = 0$, a constant.

is linear.

ex) $yy'' + ay^2 = 0$ is linear because

$y(y'' + ay) = 0$ or $y'' + ay = 0$.

ex) $yy' + a = 0$ is nonlinear.

ex) $y'' + \sin y = 0$ is nonlinear.

ex) $y'' + byy' + cy = 0$ is nonlinear.

ex) $y'' + \cos[x]\, y = 0$

linear equation with coefficients that depend on the independent variable. //

The "ORDER" of an ODE:

Refers to the highest derivative in the ODE.

ex) $y' = f(x,y)$ 1st order ODE

ex) $y''' + yy' = 0$

3rd order (nonlinear) ODE.

Learning how to solve 1st order ODE's:

"Separable"

"Linear"

"Exact"

"Homogeneous"

"Special"

Separable Equations
(1st order ODE's)

Consider $y = y(x)$,

$$(\ddagger) \qquad \frac{dy}{dx} = h(x)\, g(y)$$

where $h(x)$ is a function only of x.

$g(y)$ is only a function of y.

The Solution: Manipulate $(\ddagger)$ to obtain

$$\frac{dy}{g(y)} = h(x)\, dx.$$

Then integrate both sides:

$$\int \frac{dy}{g(y)} = \int h(x)\, dx$$

ex) $\frac{dy}{dx} = -xy$, compare to (‡)

here $h(x) = -x$

$g(y) = y$

ODE is "separable" first order (linear)

$$\frac{dy}{y} = -x\,dx$$

integrating both sides:

$$\int \frac{dy}{y} = -\int x\,dx$$

$$\ln|y| + c_1 = -\frac{1}{2}x^2 + c_2$$

$$\ln|y| = -\frac{1}{2}x^2 + c$$

exponentiate b.s.

$$e^{\ln|y|} = e^{-\frac{1}{2}x^2 + c}$$

$$|y| = e^{-\frac{1}{2}x^2} e^{c} = K e^{-\frac{1}{2}x^2}$$

$$y = K e^{-\frac{1}{2}x^2}$$ this is an explicit solution. //

ex) $$y' x \cos y = y(x^3 + 3)$$

$$\frac{dy}{dx} \frac{x \cos y}{y} = x^3 + 3$$

$$dy \frac{\cos y}{y} = \frac{x^3+3}{x} dx = \left(x^2 + 3/x\right) dx$$

$$dy\, h(y) = g(x)\, dx$$ separable.

So we know the method of solution:

Integrate b.s. after separating:

$$\int dy \frac{\cos y}{y} = \int g(x)\,dx$$

$$\int dy \frac{\cos y}{y} = \frac{1}{3}x^3 + 3\ln|x| + C$$

an implicit solution. In fact, the solution is not defined for $x<0$ and $y=0$. So we will also be concerned with whether a solution has a limited domain of definition.

ex) $\frac{dy}{dx} = -\frac{(x^2-1)}{x^2} y^3$

$\frac{dy}{y^3} = -\left(\frac{x^2-1}{x^2}\right)dx$ separable.

integrating b.s.

$$\int y^{-3}\,dy = \int (x^{-2}-1)\,dx$$

$$\frac{1}{2y^2} = \frac{1}{x} + x + C$$