A TRANSLATION THEOREM

If $F(s) = \mathcal{L}(f(t))$ exists for some $s > a \geq 0$ and $c > 0$ then

$$\boxed{\mathcal{L}\left[u(t-c)\,f(t-c)\right] = e^{-cs}F(s).}$$

We verify this:

$$\mathcal{L}\left(u(t-c)\,f(t-c)\right) = \int_0^\infty u(t-c)\,f(t-c)e^{-st}\,dt$$

$$= \int_c^\infty f(t-c)e^{-st}\,dt \qquad \text{let } u = t-c \;\therefore\; t = u+c$$

$$du = dt$$

$$= \int_0^\infty f(u)\,e^{-s(u+c)}\,du = e^{-cs}\int_0^\infty f(u)e^{-su}\,du$$

$$= e^{-cs}F(s) \quad //$$

Can show that

$$(*)\quad \mathcal{L}\big(\mathcal{U}(t-c)\,f(t)\big) = e^{-cs}\,\mathcal{L}\big(f(t+c)\big)$$

ex) $\mathcal{L}^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right)$

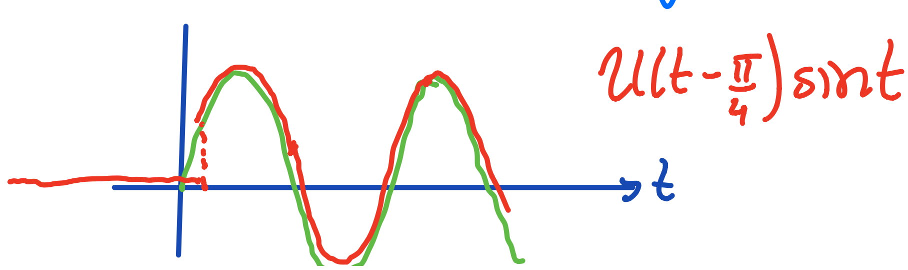
from table
$\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$

so: $$\mathcal{U}(t-\pi)\sin(t-\pi) = \begin{cases} 0 & 0 \le t < \pi \\ \sin(t-\pi) & t \geq \pi \end{cases}$$

Note: $\sin(t-\pi) = -\sin t$

$$\therefore\ \mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) = -\mathcal{U}(t-\pi)\sin t$$

ex) $\mathcal{L}(g(t)) = \mathcal{L}\left(\mathcal{U}\left(t-\frac{\pi}{4}\right)\sin t\right)$



Can use (A) but

$$\mathcal{L}(g(t)) = \mathcal{L}\left(\mathcal{U}(t - \tfrac{\pi}{4})\sin t\right)$$

$$= e^{-\frac{\pi}{4}s}\,\mathcal{L}\left(\sin\left(t + \tfrac{\pi}{4}\right)\right) \quad \text{using } A$$

$$= e^{-\frac{\pi s}{4}}\,\mathcal{L}\left(\frac{1}{2i}e^{i(t+\frac{\pi}{4})} - \frac{1}{2i}e^{-i(t+\frac{\pi}{4})}\right)$$

$$= \frac{e^{-\frac{\pi s}{4}}}{2i}\,\mathcal{L}\left(e^{it}e^{i\pi/4} - e^{-it}e^{-i\pi/4}\right)$$

$$= \frac{e^{-\frac{\pi s}{4}}}{2i}\left[e^{i\pi/4}\,\mathcal{L}(e^{it}) - e^{-i\pi/4}\,\mathcal{L}(e^{-it})\right]$$

from: Table $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$$= \frac{e^{-\frac{\pi s}{4}}}{2i}\left[e^{i\pi/4}\frac{1}{s-i} - e^{-i\pi/4}\frac{1}{s+i}\right]$$

$$= \frac{e^{-\frac{\pi s}{4}}}{2i}\left[\frac{1}{s-i}\frac{\sqrt{2}}{2} - \frac{1}{s+i}\frac{\sqrt{2}}{2}\right]$$

$$= \frac{e^{-\frac{\pi s}{4}}}{2i}\frac{\sqrt{2}}{2}\left[\frac{s+i-s+i}{s^2+1}\right] = \frac{e^{-\frac{\pi s}{4}}}{2i}\frac{\sqrt{2}}{2}\left[\frac{2i}{s^2+1}\right]$$

$$= \frac{e^{-\pi s/4}\sqrt{2}}{2}\left[\frac{1}{s^2+1}\right]$$

ANOTHER TRANSLATION THEOREM

If $F(s) = \mathcal{L}(f(t))$

$$\mathcal{L}\left(e^{at}f(t)\right) = F(s-a)$$

We verify:

$$\mathcal{L}\left(e^{at}f(t)\right) = \int_0^\infty e^{at}f(t)e^{-st}\,dt$$

$$= \int_0^\infty f(t)\,e^{(a-s)t}\,dt$$

let $(a-s)t = -\delta\xi$

$\delta = s-a$

$$= \int_0^\infty f(\xi)e^{-\delta\xi}d\xi$$

This is a laplace transform in the δ domain

$$\int_0^\infty f(\xi)e^{-\delta\xi}d\xi = F(\delta)$$

$$= F(s-a)$$

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

//

ex) $\mathcal{L}(e^{3t}\sin t) = F(s-3)$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$$

$$F(s) = \frac{1}{s^2+1} \qquad \therefore F(s-3) = \frac{1}{(s-3)^2+1} \quad //$$

LAPLACE TRANSFORM OF DERIVATIVE:

$$\mathcal{L}(f'(t)) = \int_0^\infty f' e^{-st}\,dt$$

IBP : $\quad u = e^{-st} \qquad v = f$

$$du = -s\,e^{-st}\,dt \qquad dv = \frac{df}{dt}\,dt$$

$$\int_0^\infty f' e^{-st}\,dt = e^{-st} f \Big|_0^\infty + s\int_0^\infty f e^{-st}\,dt$$

$$= \lim_{R\to\infty} e^{-sR} f - e^0 f(0) + s\int_0^\infty f e^{-st}\,dt \text{ . But}$$

if f grows no faster than e^{at}

then $\lim_{t \to \infty} e^{-st} f = 0$ provided $a < s$.

$$\therefore \int_0^\infty f' e^{-st}\, dt = s\, F(s) - f(0)$$