LAPLACE TRANSFORM 7.1 - 7.2

$$\mathcal{L}(f(t)) = F(s)$$ Laplace Transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$ Inverse Laplace Transform

For a piece-wise continuous function $f(t)$

with $t \geq 0$:

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{R \to \infty} \int_0^R e^{-st} f(t)\,dt$$

THERE ARE MANY TRANSFORMS: Fourier, Hankel, etc.

All transforms $Y(z) = \int K(t,z)\,y(t)$ obey linear superposition.

LINEAR SUPERPOSITION:

if $F(s) = \mathcal{L}(f(t))$ & $G(s) = \mathcal{L}(g(t))$

then $\mathcal{L}(f(t)+g(t)) = F(s) + G(s)$

//

$\mathcal{L}(cf(t)) = cF(s)$ "Scaling"

c is a number

//

The Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}\,ds$$

Usually very challenging. Here we learn to find $\mathcal{L}^{-1}$ by properties and known transform.

//

ex) Compute the $\mathcal{L}(1)$

$$\mathcal{L}(1) = \int_0^\infty 1 e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_{t=0}^{\infty} = \frac{1}{s}$$

$$= \lim_{R\to\infty}\left(-\frac{1}{s}e^{-sR}\right) + \frac{1}{s}$$

//

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{(a-s)t}\,dt$$

a is a constant

$$= \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty = \frac{1}{s-a}$$

where $s > a$

$$= \lim_{R\to\infty} \frac{1}{a-s}e^{(a-s)R} - \frac{e^0}{a-s}$$

$$\mathcal{L}\left[\cos\omega t\right] = \int_0^\infty \cos\omega t\, e^{-st}\,dt$$

ω is a number

Use Euler's formula

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$\cosh\omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$$

$$\sinh\omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$$

$$\int_0^\infty \cos\omega t\, e^{-st}\,dt = \frac{1}{2}\int_0^\infty e^{i\omega t} e^{-st}\,dt$$

$$+ \frac{1}{2}\int_0^\infty e^{-i\omega t} e^{-st}\,dt$$

$$= \frac{1}{2}\left\{\int_0^\infty e^{(i\omega - s)t}\,dt + \int_0^\infty e^{(-i\omega - s)t}\,dt\right\}$$

$$= \frac{1}{2}\,\frac{1}{i\omega - s}\, e^{(i\omega - s)t}\Big|_0^{\infty} + \frac{1}{2}\,\frac{1}{-i\omega - s}\, e^{-(i\omega + s)t}\Big|_0^{\infty}$$

$$= -\frac{1}{2}\,\frac{1}{i\omega - s} - \frac{1}{2}\,\frac{-1}{i\omega + s}$$

$$= \frac{1}{2}\,\frac{1}{s - i\omega} + \frac{1}{2}\,\frac{1}{s + i\omega}$$

$$= \frac{1}{2}\,\frac{1}{s - i\omega}\,\frac{s + i\omega}{s + i\omega} + \frac{1}{2}\,\frac{s - i\omega}{s - i\omega}\,\frac{1}{s + i\omega}$$

$$= \frac{1}{2}\left\{\frac{s + i\omega}{s^2 + \omega^2} + \frac{s - i\omega}{s^2 + \omega^2}\right\} = \frac{s}{s^2 + \omega^2} \quad //$$

$s \neq i\omega$

$$\mathcal{L}(t) = \underset{\text{IBP}}{\int_0^{\infty} t e^{-st}\, dt} = \lim_{R \to \infty}\left(\cancelto{0}{-\frac{t}{s} e^{-st}\Big|_0^R} + \frac{1}{s}\int_0^R e^{-st}\, dt \right)$$

$$= \frac{1}{s}\mathcal{L}(1) = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2} \quad //$$

ex) $\mathcal{L}(t^p)$

Use Gamma Function

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx$$

$\Gamma(p+1) = p!$ ← factorial

if p is an integer

$$\mathcal{L}(t^p) = \int_0^\infty t^p e^{-st}\,dt$$

let $x = st$ $\quad t = x/s$

$dx = s\,dt \quad dt = \frac{dx}{s}$

$$\mathcal{L}(t^p) = \int_0^\infty \left(\frac{x}{s}\right)^p e^{-x}\,\frac{dx}{s} = \frac{1}{s^{p+1}}\int_0^\infty x^p e^{-x}\,dx$$

$$= \frac{1}{s^{p+1}}\Gamma(p+1)$$

if p is natural # $\quad \frac{1}{s^{p+1}}p!$

Use induction:

for $p=0$ $\mathcal{L}(t^p) = 1$

for $p \geq 1$ $\int_0^\infty x^p e^{-x} dx = -x^p e^{-x}\Big|_0^\infty + p\int_0^\infty x^{p-1} e^{-x} dx$

IBP

$= p\,\Gamma(p)$

$\therefore$ $\Gamma(p+1) = p\,\Gamma(p)$ $p=0,1,2\ldots$

with $p=0$ $\Gamma(1)=1$

$$\mathcal{L}(t^p) = \frac{p!}{s^{p+1}} \quad p=0,1,2\ldots \quad s>0$$