FORCED OSCILLATIONS & RESONANCE

$$(\$) \quad y'' + 2\lambda y' + \omega^2 y = f_0 \cos \omega_0 t$$

ω_0 is the "driving frequency"

f_0 is the external drive amplitude

THINK OF f_0, ω_0 as parameters in the following.

The solution of $(\$)$ is $y = y_H + y_P$

ASSUME THAT λ, m, k are such that

$$y_H = e^{-\lambda t}\left[A \cos \omega_d t + B \sin \omega_d t\right]$$

Damped freq: $\omega_d = \sqrt{\omega^2 - \lambda^2}$ $\quad \omega^2 \geq \lambda^2$

$0 \leq \lambda$ damping

Note that $y_H \to 0$ as $t \to \infty$, iff $\lambda > 0$.

By MUC, we know that the particular solution will be

$$y_p = C\cos\omega_0 t + D\sin\omega_0 t$$

$$= R\cos(\omega_0 t + \delta)$$

$R = \sqrt{C^2 + D^2}$ $\tan\delta = \frac{-D}{C}$. Solving

for

$$R = \frac{f_0}{\Delta}$$

$$\Delta = \sqrt{m^2(\omega^2 - \omega_0^2) + \gamma^2\omega_0}$$

mass

Let's leave f_0 fixed at some value.

Vary ω_0 and examine what happens to $y(t)$. Leave f_0 fixed at some value.

Two cases:

$\gamma = 0$ $y(t) = y_H(t) + y_P(t)$

$\gamma > 0$ $y(t) = y_P$, for t really large. That is

$$\lim_{t\to\infty} y(t) = y_p(t) = \frac{f_0}{\Delta}\cos(\omega_0 t + \delta)$$

$$\cos\delta = \frac{m(\omega^2 - \omega_0^2)}{\Delta} \qquad \sin\delta = \frac{\gamma\omega_0}{\Delta}$$

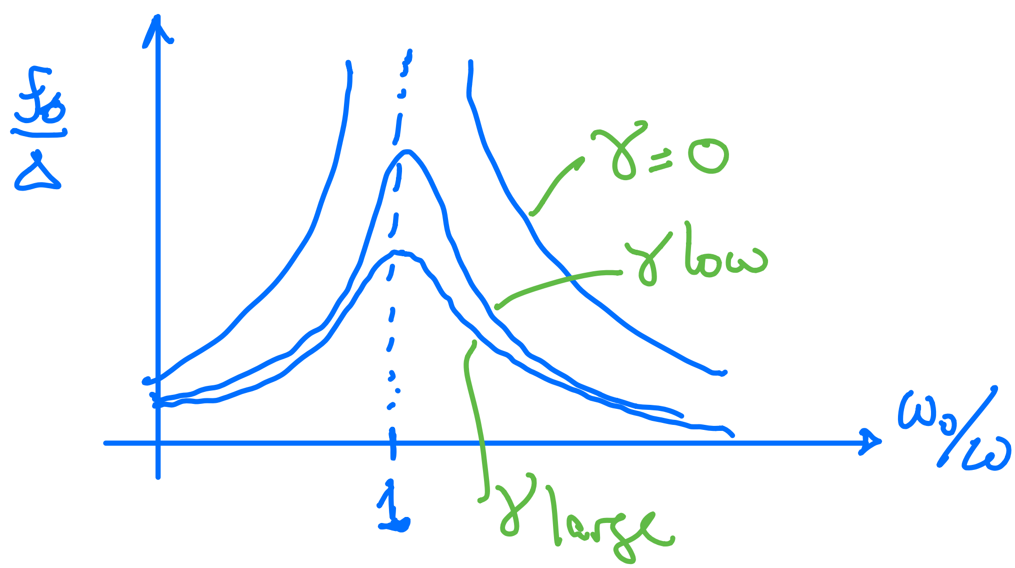
$$\Delta = \sqrt{m^2(\omega - \omega_0^2) + \gamma^2\omega_0}$$

if $\gamma = 0$ no damping

$$\lim \frac{f_0}{\Delta} \nearrow \infty \quad \text{if } \omega_0 = \omega$$

WE CALL THIS RESONANCE

if $\gamma \neq 0$ $\qquad \frac{f_0}{\Delta} > f_0$



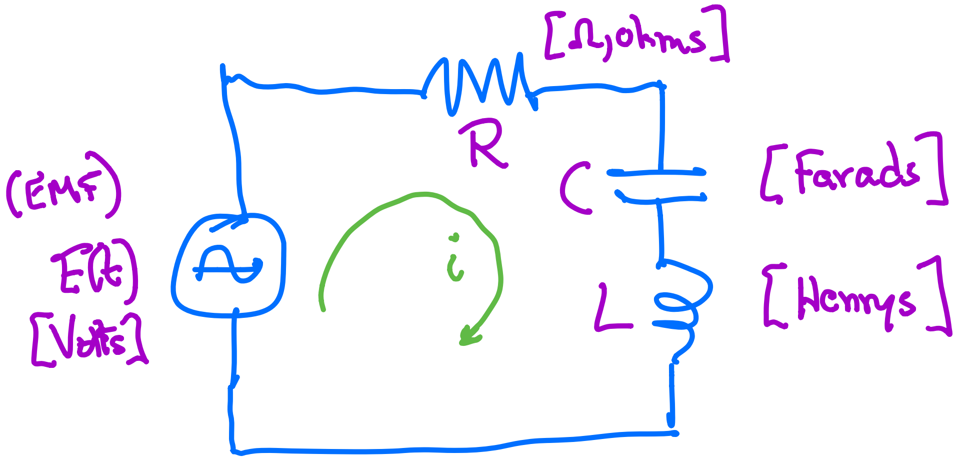
RLC CIRCUITS

(SHO simple harmonic oscillators)

ODE $$y'' + \beta y' + \omega^2 y = f(t) \qquad t>0$$

I.C. $$y(0) = y_0 \qquad y'(0) = y_1$$

MECHANICAL	RLC CIRCUIT	
$\beta = \frac{\gamma}{m} = 2\lambda$	$2\lambda = \beta = \frac{R}{L}$	DAMPING
$\omega^2 = \frac{k}{m}$	$\omega^2 = \frac{1}{LC}$	NATURAL FREQUENCY
$f(t) = \frac{F(t)}{m}$	$f(t) = \frac{E(t)}{L}$	FORCING
$y = u$ displacement	$y = Q$ charge	MEANING OF y
$y(0)$ initial displacement	$y(0)$ initial charge	
$y'(0)$ initial velocity	$y'(0)$ initial current	



$i = \frac{dq}{dt}$ current [Amperes]

CONSERVATION OF ENERGY (KIRKHOFF'S LAW)

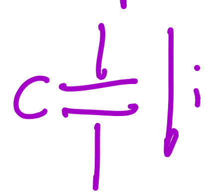
yields 2nd order linear ODE:

E(t) known voltage EMF

$V_R = iR$ voltage drop through resistor

$V_L = L\frac{di}{dt}$ voltage drop through inductor

$V_c = \frac{Q}{C}$ voltage drop through capacitor



$$\frac{dV_c}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{1}{C} i$$

$$E(t) = iR + \frac{Q}{C} + L\frac{di}{dt}$$ since $i = \frac{dQ}{dt}$

then

$$E(t) = R\frac{dQ}{dt} + \frac{Q}{C} + L\frac{d^2Q}{dt^2} \qquad (*)$$

$(*)$ is of the form ($):

$$\underbrace{\frac{E(t)}{L}}_{f(t)} = \underbrace{\frac{R}{L}}_{\beta}\frac{dQ}{dt} + \underbrace{\frac{1}{LC}}_{\omega^2}Q + \frac{d^2Q}{dt^2}$$

LAPLACE TRANSFORMS 7.1~7.3

Useful in solving linear systems of ODE's.

Especially useful for high order differential equations

We'll use $\mathcal{L}$ to indicate the application of a Laplace transform. We'll use $\mathcal{L}^{-1}$ to indicate the application of the inverse or "undo" transform.

let $L = a_n(t)\dfrac{d^{(n)}}{dt^{(n)}} + a_{n-1}\dfrac{d^{(n-1)}}{dt^{(n-1)}} + \dots$

$$a_1 \frac{d}{dt} + a_0$$

the symbol for the n^{th} order linear differential operator

Use Laplace transforms to solve

$$\boxed{\begin{array}{l} Ly(t) = g(t) \quad \text{ODE} \\ \text{for } \; y = y(t) \quad t \geq 0 \\ \text{with} \\ y(0) = y_0,\ y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1} \quad \text{I.C.} \end{array}}$$

If we apply Laplace transform to $y(t)$, we indicate it as:

$$Y(s) = \mathcal{L}(y(t))$$

The inverse is then

$$y(t) = \mathcal{L}^{-1}(Y(s))$$

The Laplace Transform applies to any piecewise continuous function $f(t)$ $t \geq 0$, such

$$|f(t)| \leq K e^{at} \qquad t > M \geq 0, \quad a > 0$$

(there's also an important condition to be satisfied for $f(t)$ near $t = 0$).