FORCED OSCILLATIONS 5.3

ODE

$$\boxed{\begin{gathered} u'' + 2\lambda u' + \omega^2 u = f(t) \\ \omega = \sqrt{\frac{k}{m}} \qquad \beta = \frac{\gamma}{m} \qquad f(t) = \frac{F(t)}{m} \\ \lambda = \frac{\beta}{2} \end{gathered}}$$

I.C. $u(0) = u_0 \qquad u'(0) = V_0$

General Solution:

$$u(t) = u_H(t) + u_p(t)$$

ex) $u'' + 2\lambda u' + \omega^2 u = \sin t$

Set $\lambda = 1 \quad \omega^2 = 3$

I.C. $u(0) = u'(0) = 0$

$$u_H + 2u_H' + 3u_H = 0$$

$$u_H = e^{mt}$$

(*) $m^2 + 2m + 3 = 0$

More generally $m^2 + 2\lambda + \omega^2 = 0$

$$m_{1,2} = -\frac{2\lambda}{2} \pm \frac{1}{2}\sqrt{4\lambda^2 - 4\omega^2}$$

$$m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2} = -\lambda \pm i\omega_d$$

For (*): $\quad m_{1,2} = -1 \pm i\sqrt{2} = -1 \pm i\omega_d$

$$u_H = e^{-t}\left\{ A\cos\sqrt{2}t + B\sin\sqrt{2}t \right\}$$

Do not apply I.C. yet!

Note that forcing frequency is 1 rad/sec

the natural frequency is $\omega = \sqrt{3}$ rad/sec

the damped frequency is $\omega_d = \sqrt{2}$ rad/sec

Find

$$u_p = C\cos t + D\sin t$$

(via the MUC)

$$u_p' = -C\sin t + D\cos t$$

$$u_p'' = -C\cos t - D\sin t = -u_p$$

Replace it in $u_p'' + 2u_p' + 3u_p = \sin t$

$$-u_p + 3u_p - 2C\sin t + 2D\cos t = \sin t$$

$$2u_p - 2C\sin t + 2D\cos t = \sin t$$

$$2C\cos t + 2D\sin t - 2C\sin t + 2D\cos t = \sin t$$

$$(2C + 2D)\cos t = 0 \qquad C = -D$$

$$(2D - 2C)\sin t = \sin t$$

$$2D + 2D = 1 \quad \text{or} \quad D = \frac{1}{4} \quad C = -\frac{1}{4}$$

$$u(t) = u_H + \frac{1}{4}\sin t - \frac{1}{4}\cos t$$

$$= e^{-t}\left[A\cos\sqrt{2}t + B\sin\sqrt{2}t\right]$$

$$+ \frac{1}{4}\sin t - \frac{1}{4}\cos t$$

Now apply I.C. $u(0) = u'(0) = 0$

$$u'(t) = -e^{-t}\left[A\cos\sqrt{2}t + B\sin\sqrt{2}t\right]$$

$$+ e^{-t}\left[-\sqrt{2}A\sin\sqrt{2}t + \sqrt{2}B\cos\sqrt{2}t\right]$$

$$+ \frac{1}{4}\cos t + \frac{1}{4}\sin t$$

$$u(0) = 0 = A - \frac{1}{4} \Rightarrow A = \frac{1}{4}$$

$$u'(0) = 0 = -A + \sqrt{2}B + \frac{1}{4} = \sqrt{2}B$$

$$\therefore B = 0$$

$$u(t) = \underbrace{e^{-t}\frac{1}{4}\cos\sqrt{2}t}_{\text{transient oscillation}} + \underbrace{\frac{1}{4}\sin t - \frac{1}{4}\cos t}_{\text{steady state}}$$

Steady state: is defined as

$$\lim_{t\to\infty} u(t)$$

In this case

$$\lim_{t\to\infty} u(t) = \frac{1}{4}(\sin t - \cos t) = \frac{\sqrt{2}}{4}\cos(t+\phi)$$

$$\phi = \arctan\left(\frac{-1}{1}\right) = -\arctan(1)$$

$$= -\frac{\pi}{4}$$

RESONANCE

$$y'' + 2\lambda y' + \omega^2 y = f_0 \cos\omega_0 t$$

$$y_H = e^{-\lambda t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

$$\omega_d = \sqrt{\omega^2 - \lambda} \qquad \omega^2 \geq \lambda > 0$$

$$\lim_{t\to\infty} y_H = 0$$