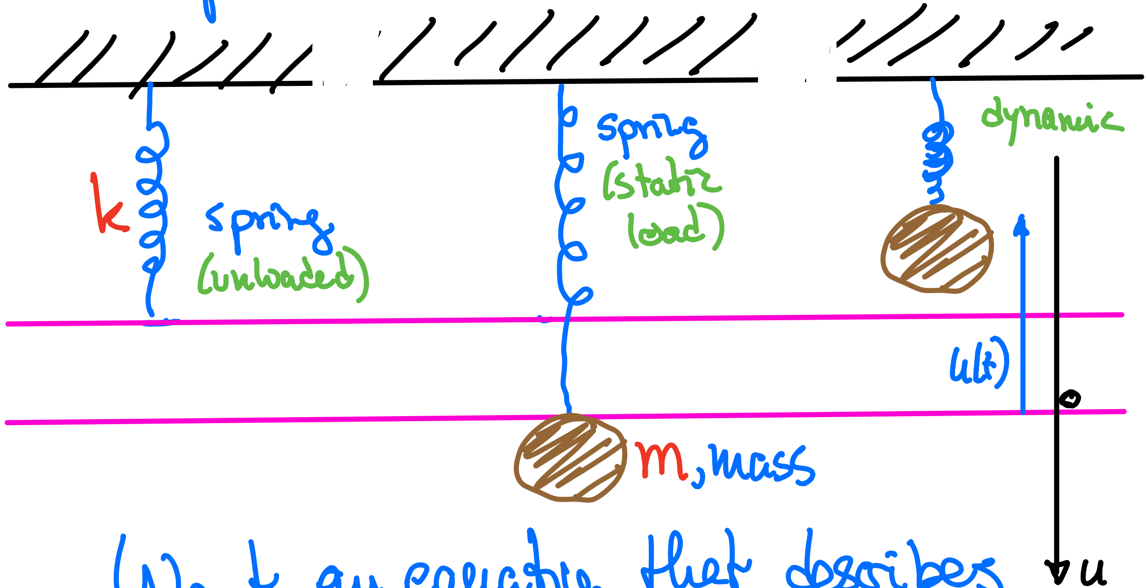
MECHANICAL VIBRATIONS Ch 5.1 & 5.2

Simple Harmonic Motion



Want an equation that describes the displacement $u(t)$ as a function of t, time.

$[u(t)]$ = length

$[t]$ = time

$[m]$ = mass (slugs in imperial units)

Use Newton's 2nd Law

$$\underbrace{m\,u''(t)}_{\text{acceleration}} = mg - k\left[u(t) + L\right]$$

Pictured as above $u>0$ would be downward

mg is gravitational force

$k[u(t)+L]$ is the "spring" force

(includes static & dynamic)

k is the spring constant

force of gravity

$(\star)$ $\quad mg = kL \quad$ static force balance

is the force balance when $u(t)=0$

Back to Newton's Law, the remaining balance of forces is:

$$m u''(t) = -k u(t)$$

dynamic force balance

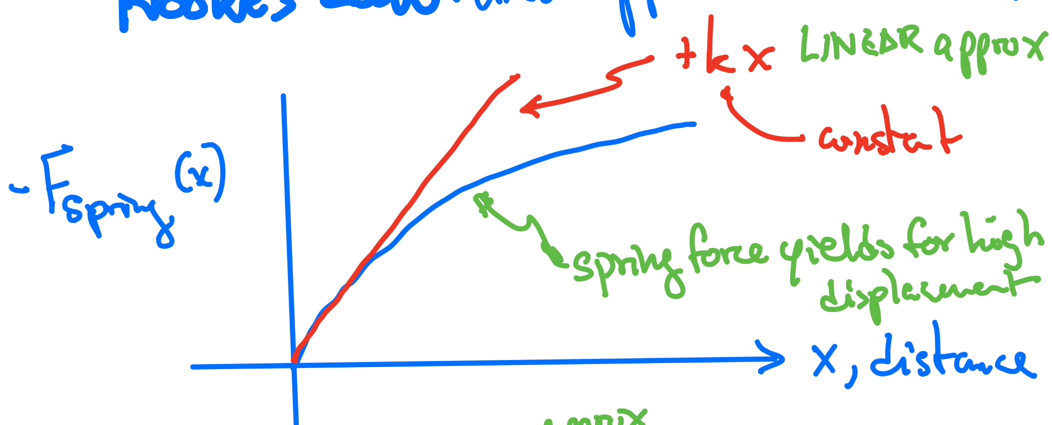
We can use $(\star)$ to find

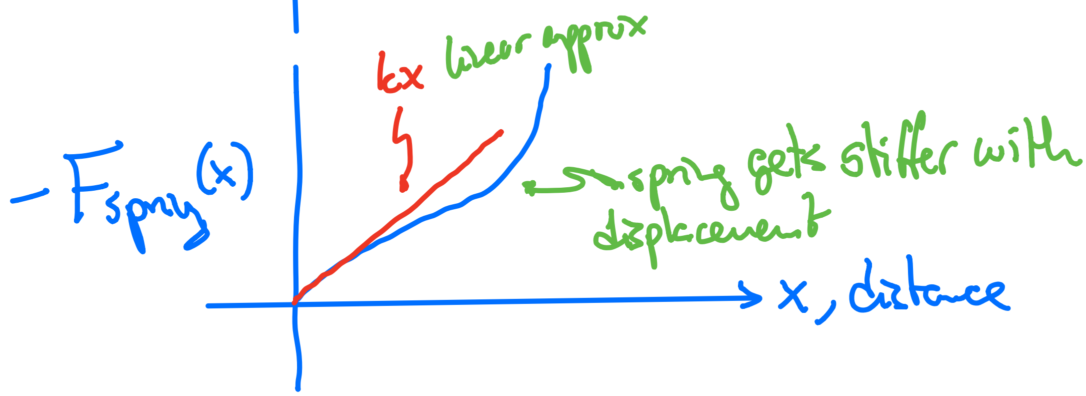
$$\boxed{k = \frac{mg}{L}}$$

$$[k] = \frac{\text{force}}{\text{length}}$$

$F_{spring} \approx -k\,u(t)$ is a LINEAR APPROXIMATION TO SPRING FORCE

Hooke's Law: linear approximation to F_{spring}





We're approximating spring force by a linear relationship (ok for small displacements, $u(t)$)

The dynamic balance is:

$$\therefore\; m\,u''(t) + k\,u(t) = 0$$

OR

$(\ddagger)$ $$\boxed{u''(t) + \omega^2 u(t) = 0}$$

$$\omega = \sqrt{\frac{k}{m}}$$ angular frequency

$$[\omega] = \frac{(\text{rad})}{\text{time}}$$

$$T = \frac{2\pi}{\omega}$$ Period of oscillation

frequency $$\nu = \frac{\omega}{2\pi}$$ Hertz "cycles per second"

SOLUTION OF $(\ddagger)$: linear 2nd order C.C.

let $u \propto e^{lt}$ substitute into $(\ddagger)$

then $$(l^2 + \omega^2)e^{lt} = 0$$

$l_{1,2} = \pm i\omega$ are the roots

$$u(t) = A\cos\omega t + B\sin\omega t$$

SINUSOIDAL MOTION

A, B are constants, determined by initial conditions.

FORCED, DAMPED, SIMPLE HARMONIC MOTION:

Add DASHPOT (DAMPER) & EXTERNAL FORCE:

$F_s = -k\,u(t) - kL$

$F_g = mg$

$F_d(t) = -\gamma u'(t)$

$$\underbrace{m\,u''(t)}_{\text{inertial force}} = \underbrace{k\,u(t)}_{\text{spring}} - \underbrace{\gamma\,u'(t)}_{\text{shock (a linear approximation)}} + \underbrace{F(t)}_{\text{motive force}}$$

$$u'' + \beta u' + \omega^2 u = f(t)$$

$$\beta = \frac{\gamma}{m} \qquad \omega^2 = \frac{k}{m} \qquad f(t) = \frac{F(t)}{m}$$

Non-homogeneous 2nd order const coefficient linear ODE

$$u(t) = u_H(t) + u_p(t)$$

The initial conditions

$u(0) = u_0$ initial displacement

$u'(0) = V_0$ initial velocity

DAMPED SIMPLE HARMONIC MOTION:

Focus on $f(t) = 0$ for now:

$$u'' + \beta u' + \omega^2 u = 0$$

$$u = u_H(t) = e^{\ell t}$$

$$\ell^2 + \beta \ell + \omega^2 = 0$$

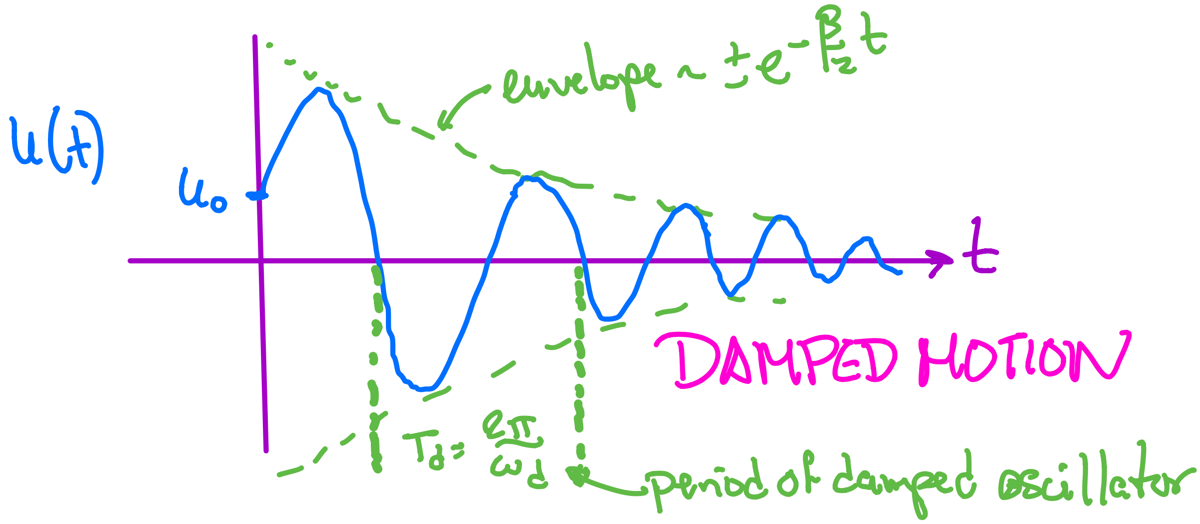
$$\lambda_{1,2} = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 - 4\omega^2}$$

$$= -\frac{\beta}{2} \pm \omega_d$$

If $4\omega^2 > \beta^2$

$$u(t) = \left(A\cos\omega_d t + B\sin\omega_d t\right)e^{-\frac{\beta}{2}t}$$

i.e. the roots are complex conjugates



DAMPED MOTION

N.B. $\beta = 0$

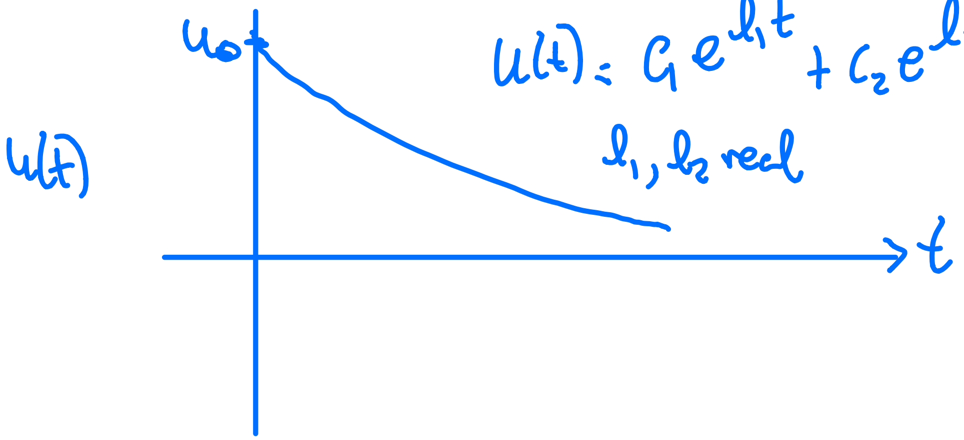
$$u(t) = A\cos\omega t + B\sin\omega t$$

recover undamped case

If $\beta^2 > 4\omega^2$ OVERDAMPED MOTION

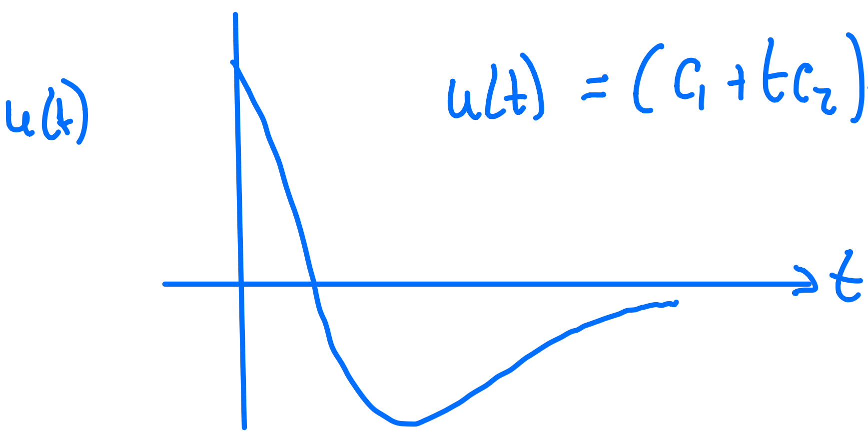
$$u(t) = c_1 e^{l_1 t} + c_2 e^{l_2 t}$$

l_1, l_2 real



If $\beta^2 = 4\omega^2$ CRITICALLY DAMPED MOTION

$$u(t) = (c_1 + t c_2) e^{-\frac{\beta}{2} t}$$



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LINEAR COMBINATION OF SIN/COS

Often, we encounter

$$y = a\cos\beta t + b\sin\beta t$$

Which can be expressed as

I
$$\begin{cases} y = D\sin(\beta t + \phi) \\ D = \sqrt{a^2+b^2} \quad \phi = \arctan\left(\frac{a}{b}\right) \end{cases}$$

Alternatively

II
$$\begin{cases} y = D\cos(\beta t + \theta) \\ D = \sqrt{a^2+b^2} \ ; \ \theta = \arctan\left(-\frac{b}{a}\right) \end{cases}$$

(I) follows from $\sin(p+q) = \sin p\cos q + \cos p\sin q$

(II) follows from $\cos(p+q) = \cos p\cos q - \sin p\sin q$

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ex) Mass 1 slug suspended from a spring, $k = 9$ lb/ft

$u(0) = 1$ ft above equil

$u'(0) = -3$ ft/s

Find times t^* at which the mass moves

(downwards) at $u'(t^*) = 3$ ft/sec

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = 3 \text{ rad/sec}$$

$$u'' + \omega^2 u = 0 \quad \text{or} \quad u'' + 9u = 0$$

has a solution

$$u(t) = A\cos 3t + B\sin 3t$$

I.C. $u(0) = -1$ $\quad u'(0) = 3$

$$u(0) = -1 = A$$

$$u'(t) = -3A\sin 3t + B3\cos 3t$$

$$u'(0) = -3 = 3B \quad \Rightarrow B = -1$$

$$\therefore \ u(t) = -\cos 3t - \sin 3t$$

We can then write this as

$$u(t) = D\cos(3t + \phi)$$

$$D = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\tan\phi = \left(\frac{-1}{1}\right) = -1$$

$$\phi = \frac{-\pi}{4}$$

$$u(t) = \sqrt{2}\cos\left(3t - \frac{\pi}{4}\right)$$

Now, we need to find the times $\hat{t}^*$ when $u'(\hat{t}^*) = 3$ ft/sec

$$u'(t) = -3\sqrt{2}\sin\left(3t - \frac{\pi}{4}\right)$$

$$3 = -3\sqrt{2}\sin\left(3t^* - \frac{\pi}{4}\right)$$

$$-\frac{1}{\sqrt{2}} = \sin\left(3t^* - \frac{\pi}{4}\right),$$ t^* would be the first time $u' = 3$ ft/s

$$\arcsin\left(-\frac{1}{\sqrt{2}}\right) = 3t^* - \frac{\pi}{4}$$

$$t^* = \frac{\pi}{12} - \frac{1}{3}\arcsin\left(\frac{1}{\sqrt{2}}\right)$$

$$t^* = \frac{\pi}{12} - \frac{1}{3}\left(-\frac{\pi}{4}\right) = \frac{2}{12}\pi = \frac{\pi}{6}$$

since the motion is periodic with period $T = \frac{2\pi}{3} = \frac{4\pi}{6}$

$$\tilde{t}^* = \frac{\pi}{6} + nT \quad , n = 0, 1, 2 \ldots$$

$$\tilde{t}^* = \frac{\pi}{6}(1 + 4n) \quad , n = 0, 1, 2 \ldots$$

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