VARIATION OF PARAMETERS

Ch 4.7

In this class we focus on two types of n^{th} order linear ODE's

$$(\ddagger) \qquad \mathcal{L}\, y(x) = f(x)$$

where $\mathcal{L} = \sum_{i=0}^{n} a_i(x) \dfrac{d^{(i)}}{dx^{(i)}}$

* Constant Coefficient: $a_i(x) = a_i \quad i=0,1..,n$ are constants.

* Euler Equations: $a_i(x) = c_i x^i \quad i=0,1,2..,n$ where c_i are constants.

The solution of $(\ddagger)$ is

$$y = y_H + y_P$$

where $\begin{cases} \mathcal{L}\, y_H = 0 \\ \mathcal{L}\, y_P = f(x) \end{cases}$

To solve $\mathcal{L}y_p = f(x)$ we learned the Method of Undetermined coefficients (i.e. make a good guess and follow the consequences).
This method applies ONLY to $(\ddagger)$ of the type Constant coefficients.
We learn another method that can solve ANY $(\ddagger)$ (including constant coefficients and Euler equations).

We illustrate the method of Variation of parameters as applied to the 2nd order linear ODE:

$$(\ddagger) \quad y'' + p(x)\,y' + q(x)\,y = f(x)$$

$$y = y(x)$$

The solution to (∓) is

$$y = y_H + y_P$$

Let's assume we know the fundamental solution (y_1, y_2). These satisfy

$$\boxed{y_i'' + p(x)\, y_i' + q(x)\, y_i = 0 \qquad i = 1, 2.}$$

To obtain a solution to

$$(\cancel{*}) \quad y_p'' + p(x)\, y_p' + q(x)\, y_p = f(x)$$

assume (A) $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

take derivatives

(B) $y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$

(C) $y_p'' = u_1'' y_1 \; 2u_1' y_1' + u_1 y_1''$

$\quad + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''$

Substitute (A), (B), (C) into (*)

$$(*) \quad u_1''y_1 + 2u_1'y_1' + u_1y_1''$$
$$+ u_2''y_2 + 2u_2'y_2' + u_2y_2''$$
$$+ p(x)\left[u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'\right]$$
$$+ q(x)\left[u_1y_1 + u_2y_2\right]$$
$$= f(x)$$

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We note that

these satisfy $\mathcal{L}y_H = 0$

$$u_2\left[y_2'' + p\,y_2' + q\,y_2\right] = 0$$
$$u_1\left[y_1'' + p\,y_1' + q\,y_1\right] = 0 .$$

A little calculation: let's compute

$$\left(u_1'y_1 + u_2'y_2\right)' = u_1''y_1 + u_1'y_1' + u_2''y_2 + u_2'y_2'$$

Let's look at what remains in (*):

$$u_1''y_1 + u_1'y_1' + u_1'y_1'$$
$$+ u_2''y_2 + u_2'y_2' + u_2'y_2'$$
$$+ p(x)\left[u_1'y_1 + u_2'y_2\right] = f(x)$$

So ◯ is the derivative of

set $u_1'y_1 + u_2'y_2 = 0$ Ⓒ

so the terms ◯ equal 0

Round up the remaining terms ◯

$$u_1'y_1' + u_2'y_2' = f(x) \quad Ⓓ$$

Rewrite Ⓒ & Ⓓ

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

By Cramer's Rule:

$$u_1' = \frac{1}{W} \det \begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix} = -\frac{y_2 f(x)}{W}$$

$$(*) \quad u_2' = \frac{1}{W} \det \begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix} = \frac{y_1 f(x)}{W}$$

$$W = y_1 y_2' - y_1' y_2'$$

$$u_1 = \int u_1' \, dx$$

$$u_2 = \int u_2' \, dx$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

ex) $y'' - 2y' - 3y = -3te^{-t}$

$$y(t) = y_H(t) + y_p(t)$$

$y_H = e^{mt}$

$\mathcal{L} y_H = 0 \Rightarrow m^2 - 2m - 3 = 0$

$m_{1,2} = -1, 3$

$y_1 = e^{-t} \qquad y_2 = e^{3t}$

Using (¥) $\qquad W = e^{-t} 3e^{3t} + e^{3t} e^{-t} = 4e^{2t}$

$\mathcal{L} y_p = -3te^{-t}$

$y_p = u_1 e^{-t} + u_2 e^{3t}$

$$u_1' = \frac{-e^{3t}(-3te^{-t})}{W} = \frac{+3}{4} t$$

$$u_2' = \frac{e^{-t}(-3te^{-t})}{W} = -\frac{3}{4} te^{-4t}$$

$$u_1 = \frac{3}{4}\int t\,dt = \frac{3t^2}{8}$$

$$u_2 = -\frac{3}{4}\int t e^{-4t}\,dt = -\frac{3}{4}\left[-\frac{1}{4}te^{-4t} - \frac{1}{16}e^{-4t}\right]$$

$$= \frac{3}{16}\left[t - \frac{1}{4}\right]e^{-4t}$$

$$\therefore y_p = \frac{3t^2}{8}e^{-t} + \frac{3}{16}\left(t - \frac{1}{4}\right)e^{-t}$$

$$y = c_1 e^{-t} + c_2 e^{3t} + \frac{3}{16}\left(t - \frac{1}{4}\right)e^{-t} + \frac{3}{8}t^2 e^{-t}$$