The MUC

$$\mathcal{L} y = f(x)$$

for non-homogeneous problems

$f(x) \neq 0.$

The general solution

$$y = y_H + y_P$$

$$\begin{cases} \mathcal{L} y_H = 0 \\ \mathcal{L} y_P = f(x) \end{cases}$$

Recall there are assumed forms for

y_P

ex) $y'' + y = 4t + 10\sin t \quad (\cancel{A})$

$$y = y_H + y_p$$

$$y_H'' + y_H = 0 \qquad y_H \propto e^{mt}$$

$$e^{mt}(m^2+1) = 0 \quad \text{or} \quad m = \pm i$$

$$y_H = A\sin t + B\cos t$$

Guess

$$y_p = C_1 t + C_2 + D\cos t + E\sin t$$

$$y_p' = C_1 - D\sin t + E\cos t$$

$$y_p'' = -D\cos t - E\sin t$$

substitute into (*)

$$\underbrace{-D\cos t - E\sin t}_{y_p''} + \underbrace{D\cos t + E\sin t + C_1 t + C_2}_{y_p}$$

$$= 4t + 10\sin t$$

N.G.

[Problem is that the guess has terms that appear in the homogeneous solution]

The correct assumed guess

(A) $y_p = c_1 t + c_2 + t(D\cos t + E\sin t)$

now $D\cos t + E\sin t$ is not a linearly dependent form of y_H.

$y_p' = c_1 + [D\cos t + E\sin t] + t[-D\sin t + E\cos t]$

(B) $y_p'' = [-D\sin t + E\cos t] + [-D\sin t + E\cos t] + t[-D\cos t - E\sin t]$

Replace (A) & (B) into (*)

$-2D\sin t + 2E\cos t + t[-D\cos t - E\sin t] + c_1 t + c_2 + t[D\cos t + E\sin t] = 4t + 10\sin t$

let $c_1 = 4 \quad c_2 = 0$

so what's left:

$$-2D\sin t + 2E\cos t = 10\sin t$$

$$E = 0 \qquad D = -5$$

$$y_p = 4t + t[-5\cos t] = t[4 - 5\cos t]$$

$$y = A\cos t + B\sin t + t[4 - 5\cos t]$$

$$y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x} \quad (*)$$

$$y(0) = \frac{1}{2} \qquad y'(0) = \frac{5}{2} \qquad y''(0) = -\frac{9}{2}$$

$$y = y_H + y_P$$

$$y_H''' - 2y_H'' + y_H' = 0$$

$$y_H \propto e^{mx}$$

$$(m^3 - 2m^2 + m)e^{mx} = 0$$

$$m(m^2 - 2m + 1) = 0 \qquad m_1 = 0$$

$m(m-1)^2 = 0 \qquad m_{2,3} = 1$

$y_H = \tilde{c}_0 + (\tilde{c}_1 + \tilde{c}_2 x)e^x$

since $f(x) = 2 - 24e^x + 40e^{5x}$

$(\exists)\quad y_p = Ae^{5x} + B_1 x + (G_1 + G_2 x)e^x x^2$

Replace $(\exists)$ into $(*)$

solve for A, B_1, G_1, G_2

$A = \frac{1}{2} \quad B = 2 \quad G_1 = -12 \quad G_2 = 0$

$y = \tilde{c}_0 + (\tilde{c}_1 + \tilde{c}_2 x)e^x$

$+ \frac{1}{2}e^{5x} + 2x - 12x^3 e^x$

THE IC are applied LAST!

$y(0) = \frac{1}{2} \quad y'(0) = \frac{5}{2} \quad y''(0) = -\frac{9}{2}$

used to find $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2$

$y(0) = \frac{1}{2} = \tilde{c}_0 + \tilde{c}_1 + \frac{1}{2} \implies \tilde{c}_0 = -\tilde{c}_1$

$y = \tilde{c}_0(1 - e^x) + \tilde{c}_2 x e^x + \frac{1}{2}e^{5x} + 2x - 12x^3 e^x$

$$(\dagger)\quad y' = -\hat{c}_0 e^x + \hat{c}_2 e^x + \hat{c}_2 x e^x + \frac{5}{2} e^{5x} + 2 - 36x^2 e^x - 12x^3 e^x$$

$$y'(0) = \frac{5}{2} = -\hat{c}_0 + \hat{c}_2 + \frac{5}{2} + 2 \implies -\hat{c}_0 + \hat{c}_2 = -2$$

$$\hat{c}_2 = -2 + \hat{c}_0$$

Differentiate $(\dagger)$

$$y'' = -\hat{c}_0 e^x + \hat{c}_2 e^x + \hat{c}_2 e^x + \hat{c}_2 x e^x + \frac{25}{2} e^{5x} - 72 x e^x - 36x^2 e^x - 36x^2 e^x - 12x^3 e^x$$

$$y''(0) = -\frac{9}{2} = -\hat{c}_0 + 2\hat{c}_2 + \frac{25}{2}$$

$$\boxed{\begin{aligned} -\hat{c}_0 + 2c_2 &= -\frac{34}{2} \\ -\hat{c}_0 + c_2 &= -2 \\ \hat{c}_1 &= -\hat{c}_0 \end{aligned}}$$