Section 6.1 EULER EQUATIONS

Differential Equation (Linear)

$$(\#)\begin{cases} \mathcal{L}\,y(x) = 0 \\ \mathcal{L} = c_n x^n \dfrac{d^n}{dx^n} + c_{n-1} x^{n-1} \dfrac{d^{n-1}}{dx^{n-1}} + \cdots c_1 x \dfrac{d}{dx} + c_0 \end{cases}$$

For (#) the solution is of the form

$$y = x^m \quad //$$

ex) $x^2 y'' - 2xy' - 4y = 0$ (D)

is E.E. $y = x^m$ (A)

(B) $y' = m x^{m-1}$ $\quad y'' = m(m-1)\,x^{m-2}$ (C)

Subs (A) - (C) into (D)

$$x^2 m(m-1)x^{m-2} - 2x\,m\,x^{m-1} - 4x^m = 0$$

$$\left(m(m-1) - 2m - 4\right)x^m = 0$$

For non-trivial solution i.e. $x \neq 0$

$$m^2 - m - 2m - 4 = 0$$

$$m^2 - 3m - 4 = 0$$

$$(m+1)(m-4) = 0$$

$$\therefore \quad y_1 = x^{-1} \qquad y_2 = x^4$$

$$\therefore \; y(x) = c_1 x^{-1} + c_2 x^4$$

$$x^2 y'' - 2xy' - 4y = \mathcal{L}y$$

$$\mathcal{L} = c_n x^n \frac{d^n}{dx^n} + c_{n-1} x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots c_1 x \frac{d}{dx} + c_0$$

$n=2 \quad c_2=1, \; c_1=-2, \; c_0=-4$ //

For 2nd order E.E. Solutions

$(*) \quad c_2 x^2 y'' + c_1 x y' + c_0 y = 0$

$y \propto x^m$

(a) m are 2 real distinct roots

$$y = A x^{m_1} + B x^{m_2}$$

(b) m are complex conjugates:

$$m_{1,2} = \alpha \pm i\beta$$

$$y = x^{\alpha}\left[A\cos(\beta \ln x) + B\sin(\beta \ln x)\right]$$

(c) $m_1 = m_2 = m$ (repeated root)

$$y = (A + B\ln x)\, x^m$$

knowing that $y_1 = x^m$ is a solution

to $(\cancel{*})$ for the case of a repeated root

use $y_2 = u(x)y_1(x)$ to find

$$y_2 = (\ln x)x^m = (\ln x)y_1$$

2nd ORDER LINEAR

C.C.	E.E.
$a_2 y'' + a_1 y' + a_0 y = 0$	$c_2 x^2 y'' + c_1 x y' + c_0 y = 0$
$y = e^{mx}$	$y = x^m$
real, distinct	real distinct
$y = Ae^{m_1 x} + Be^{m_2 x}$	$y = Ax^{m_1} + Bx^{m_2}$
complex conjugate	complex conjugate
$m_{1,2} = \alpha \pm i\beta$	$m_{1,2} = \alpha \pm i\beta$
$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$	$y = x^{\alpha}[A\cos(\beta\ln x) + B\sin(\beta\ln x)]$
Repeated $m_1 = m_2 = m$	Repeated $m_1 = m_2 = m$
$y = (A + Bx)e^{mx}$	$y = (A + B\ln x)x^m$

THE GENERAL SOLUTION TO THE NON-HOMOGENEOUS LINEAR N-ORDER DIFFERENTIAL EQ

Includes the constant coefficient and Euler equations

$$\mathcal{L}\, y(x) = f(x)$$

(if $f = 0$ then Homogeneous)

The General solution

$$y = y_H(x) + y_P(x)$$

"Homogeneous" "particular"

Where

$$\mathcal{L}\, y_H = 0$$

(+) $$\mathcal{L}\, y_P = f(x)$$

To find solutions to (+)

$\begin{cases} \text{MUC (Guess)} \\ \text{Variation of Parameters} \end{cases}$

4.4 Method of Undetermined Coefficients (MUC)

Guess:

$$\mathcal{L} y_p = f(x)$$

Given $f(x)$	Assumed y_p form
$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$ (a_i's are constant)	$x^s(A_n x^n + A_{n-1} x^{n-1} + \dots A_1 x + A_0)$
$P_n(x) e^{ax}$	$x^s(A_n x^n + A_{n-1} x^{n-1} + \dots A_1 x + A_0) e^{ax}$
$P_n(x) e^{ax} \begin{Bmatrix} \sin \beta x \\ \cos \beta x \end{Bmatrix}$	$E(x)$

$$E(x) = x^s\left(A_n x^n + A_{n-1}x^{n-1} + \dots A_1 x + A_0\right)e^{ax}\sin\beta x$$
$$+ x^s\left(B_n x^n + B_{n-1}x^{n-1} + \dots B_1 x + B_0\right)e^{ax}\cos\beta x$$

s in the x^s is an $0, 1, 2 \dots$, this will ensure that no term in $y_p(x)$ is a fundamental solution to $\mathcal{L}y_H = 0$.

ex) $y'' + 6y' + 9y = 3e^{2t}$

$$y = y(t)$$

$$\mathcal{L}y(t) = f(t)$$

$$y = y_H(t) + y_p(t)$$

Note that $\mathcal{L}(y_H + y_p) = \underbrace{\mathcal{L}y_H}_{=0} + \mathcal{L}y_p = f(t)$

$$= \mathcal{L}y_p = f(t)$$

① Solve $\mathcal{L}y_H = 0$

$$y_H'' + 6y_H' + 9y_H = 0$$

$$y_H \propto e^{mt}$$

$$(m^2 + 6m + 9)e^{mt} = 0$$

$m = -3$ repeated.

$$y_H = (C + Dt)e^{-3t}$$

② Solve $\mathcal{L}y_p = f(t) = 3e^{2t}$

$$y_p'' + 6y_p' + 9y_p = 3e^{2t}$$

let $y_p = Ae^{2t}$

$$y_p' = 2Ae^{2t} \quad y_p'' = 4Ae^{2t}$$

$$4Ae^{2t} + 6\cdot 2Ae^{2t} + 9Ae^{2t} = 3e^{2t}$$

$$A = \frac{3}{25}$$

$$y_p = \frac{3}{25} e^{2t}$$

$$y = (C + Dt)e^{-3t} + \frac{3}{25} e^{2t}$$ //