LINEAR INDEPENDENCE: Given functions $y_1, y_2, \dots, y_n$ where $y_i(x)$ defined on same x interval $I = (\alpha, \beta)$,

we say $\{y_i(x)\}_{i=1}^{n}$ is a LI set iff

$$c_1 y_1 + c_2 y_2 + \dots c_n y_n = 0 \text{ over } I$$

is satisfied when $c_i = 0 \quad i = 1, 2, \dots, n.$

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ex) $$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ over } I$$

if $c_1 = c_2 = 0$ then y_1 & y_2 are LI

Suppose c_1 & $c_2 \neq 0$:

$$c_1 y_1 = -c_2 y_2$$

$$y_1 = -\frac{c_2}{c_1} y_2 = \alpha y_2$$

where α is a constant.

y_2 and y_1 are, to within a constant, the same function (same structure).

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Back to linear $n^{\underline{th}}$ order ODE's:

(ODE) $\mathcal{L}\, y = 0$ with $y = y(x)$.

$\mathcal{L}$ is $n^{\underline{th}}$ order "differential operator". (ODE)

has a solution $y(x) = c_1 y_1 + c_2 y_2 \cdots c_n y_n$,

where y_i's [are linearly independent] solutions

satisfying

$$\mathcal{L} y_i = 0.$$

ex) $y_1 = d_1 \cos \beta x \qquad y_2 = d_2 \sin \beta x$

d_1 & d_2 are constants, $\beta \neq 0$ constant.

Determine whether they are LI over x interval

$I = (-\pi, \pi)$: Form

$$(*) \quad c_1\, d_1 \cos \beta x + c_2\, d_2 \sin \beta x \stackrel{?}{=} 0$$

Can we find a c_1 & $c_2 \neq 0$ that makes

(*) true?

Absorb the constants d_1 & d_2 into $\tilde{c}_1 = c_1 d_1$ & $\tilde{c}_2 = c_2 d_2$

then ($) $\tilde{C}_1 \cos\beta x + \tilde{C}_2 \sin\beta x = 0$. Suppose

$\tilde{c}_1$ & $\tilde{c}_2$ not zero:

$$\cos\beta x = -\frac{\tilde{C}_2}{\tilde{C}_1} \sin\beta x$$

No, can't be done for all x in I.

($) can only be satisfied iff $C_1 = C_2 = 0$.

in fact, for these functions it is true for all x:

$\cos\beta x$ & $\sin\beta x$ are L.I over $(-\infty, \infty)$ //

Try the Wronskian:

$$W(y_1, y_2) = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \det \begin{bmatrix} d_1 \cos\beta x & d_2 \sin\beta x \\ -\beta d_1 \sin\beta x & d_2 \beta \cos\beta x \end{bmatrix}$$

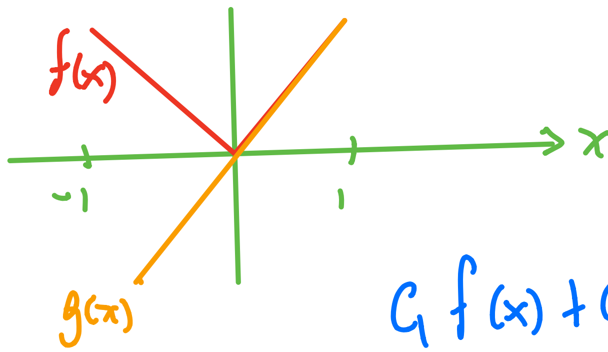
$$= d_1 d_2 \left[\cos^2\beta x + \sin^2\beta x\right]\beta$$

$$= d_1 d_2 \beta \neq 0 \text{ anywhere on } I.$$

ex) Consider $f(x) = |x|$

$g(x) = x$

for $I = (-1, 1)$



$$c_1 f(x) + c_2 g(x) = 0$$

What values can we assign to c_1, c_2? can they both be zero?

NO. Not over $(-1, 1)$.

ex) For ODE $y'' + \beta^2 y = 0$

(β is a constant)

The general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 & y_2 are LI.

$y_1 = \cos\beta x \quad y_2 = \sin\beta x$

are the FUNDAMENTAL SOLUTIONS

How do we find y_1 & y_2?

SOLUTION TO N^{th} order Linear ODE's with constant coefficients

$$(\ddagger) \quad \mathcal{L}y = 0$$

$$\mathcal{L} = a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots a_1 \frac{d}{dx} + a_0$$

where a_i are all constant!

ex) $$y'' + \beta^2 y = 0$$

$$a_2 = 1 \quad a_1 = 0 \quad a_0 = \beta^2 \quad //$$

The solution to $(\ddagger)$ is

of the form $y(x) = e^{mx}$ $(\exists)$
m is constant

Consider $y'' + \alpha y' + \beta^2 y = 0$ $(\forall)$

α & β^2 are constants.

Replace $(\exists)$ into $(\forall)$

$$m^2 e^{mx} + \alpha m e^{mx} + \beta^2 e^{mx} = 0$$

$$e^{mx}\left(m^2 + \alpha m + \beta^2\right) = 0$$

Since $e^{mx} \neq 0$ over $\mathbb{R}$

$$\Rightarrow \quad m^2 + \alpha m + \beta^2 = 0$$

$$m_{1,2} = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta^2}$$

$$\therefore \; y(x) = c_1 e^{-\frac{\alpha}{2}x} e^{+\frac{1}{2}\sqrt{\alpha^2 - 4\beta^2}\,x} + c_2 e^{-\frac{\alpha}{2}x} e^{-\frac{1}{2}\sqrt{\alpha^2 - 4\beta^2}\,x}$$

assuming $\alpha^2 \neq 4\beta^2$

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ex) $\begin{cases} \delta y'' + \omega^2 y = 0 \\ y(0) = 1 \quad y'(0) = 0 \end{cases}$ δ, ω^2 are constants.

Solve IVP

$$y'' + \bar{\omega}^2 y = 0 \qquad \bar{\omega}^2 = \frac{\omega^2}{\delta}$$

$y \propto e^{mx}$

$y(m^2 + \bar{\omega}^2) = 0$

or $m^2 = \bar{\omega}^2$ $\quad m = \pm i \bar{\omega}$

$y = c_1 e^{i\bar{\omega}x} + c_2 e^{-i\bar{\omega}x}$