6.6 The Convolution Operator

If $f(t)$ & $g(t)$ are piecewise continuous on $[0,\infty)$ the CONVOLUTION OF f AND g is

$$h(t) \equiv f * g = \int_0^t g(\tau) f(t-\tau)\, d\tau$$

convolution symbol

Notice that $f * g = h(t)$, is a function of t

The convolution is linear & satisfies associative properties and commutation:

Let $\ell(t)$ be another piecewise continuous function on $[0,\infty)$:

$$\begin{cases} f*(g+\ell) = f*g + f*\ell \\ f*(cg) = c\, f*g \\ f*g = g*f \\ f*(g*\ell) = (f*g)*\ell \end{cases}$$

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ex) $f*1 = 1*f$

$$f*1 = \int_0^t 1\, f(t-\tau)\,d\tau = -\int_t^0 f(u)\,du = \int_0^t f(u)\,du$$

$u = t-\tau \quad du = -d\tau$

when $\tau = 0 \quad u = t$, $\tau = t \quad u = 0$

ex) $$e^t * e^{-t} = \int_0^t e^{-\tau} e^{t-\tau}\,d\tau = e^t \int_0^t e^{-2\tau}\,d\tau$$

$$= e^t \left(-\frac{1}{2} e^{-2\tau}\right)\Big|_0^t = -\frac{1}{2} e^t \left(e^{-2t} - 1\right)$$

$$= \frac{1}{2} e^t \left(1 - e^{-2t}\right) = \frac{1}{2}\left(e^t - e^{-t}\right) = \sinh t \quad //$$

ex) Prove that $f*g = g*f$

$$\int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

let $x = t-\tau$ when $\begin{cases} \tau = 0 & x = t \\ \tau = t & x = 0 \end{cases}$

$dx = -d\tau$

$$-\int_t^0 f(t-x)\,g(x)\,dx = \int_0^t f(t-\tau)\,g(\tau)\,d\tau \quad //$$

The Convolution Theorem

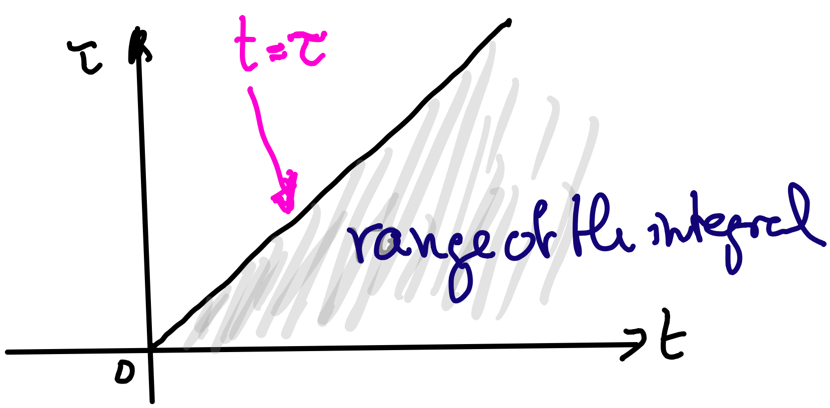
Let $f(t)$ & $g(t)$ be 2 piecewise continuous functions on $[0,\infty)$. Let $\mathcal{L}$ be the Laplace Transform operator.

$$\mathcal{L}\big(f(t)*g(t)\big) = F(s)\,G(s)$$

where $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(g(t)) = G(s)$

It's in your "Table of Laplace Transforms"

<u>Proof:</u> $\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right]dt$



Change the order of integration:

$$\int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right]d\tau$$

Let $u = t-\tau$ $\quad du = dt$ $\quad \begin{cases} t=\tau & u=0 \\ t=\infty & u=\infty \end{cases}$

$$\int_0^\infty f(\tau)\left[\int_0^\infty e^{-(u+\tau)s} g(u)\,du\right]d\tau = \int_0^\infty f(\tau)e^{-s\tau}\underbrace{\int_0^\infty e^{-su} g(u)\,du}_{G(s)}\,d\tau$$

$$= G(s)\underbrace{\int_0^\infty f(\tau)e^{-s\tau}\,d\tau}_{F(s)} = G(s)F(s)$$

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An important application in Laplace Transforms is that products of Laplace transforms can be inverse transformed as integrals in t. To see this consider:

ex) $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$, from Tables.

But it can also be found by:

$$\mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s}\right) = \mathcal{L}^{-1}\left(F(s)\,G(s)\right) = 1 * 1 = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

where $f(t)=1$ and $g(t)=1$

$$1*1 = \int_0^t d\tau = t$$

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A more non-trivial example:

ex) $$\mathcal{L}^{-1}\left(\frac{2}{(s-1)(s+2)}\right) = \mathcal{L}^{-1}\left(F(s)\,G(s)\right) = f(t)*g(t)$$

$$F(s) = \frac{2}{s-1} \qquad G(s) = \frac{1}{s+2}$$

$$\mathcal{L}^{-1}(F(s)) = 2e^{t} \qquad \mathcal{L}^{-1}(G(s)) = e^{-2t}$$

$$f(t)*g(t) = \int_0^t 2e^{-2\tau}e^{(t-\tau)}\,d\tau = 2e^{t}\int_0^t e^{-3\tau}\,d\tau$$

$$= 2e^{t}\left[-\frac{1}{3}e^{-3\tau}\right]_0^t = -\frac{2}{3}e^{-2t} + \frac{2}{3}e^{t}$$

Another Application: Helpful in solving IVP

ex) IVP $\begin{cases} y'' + y = g(t) \\ y(0) = y'(0) = 0 \end{cases}$

$$\mathcal{L}(y(t)) = Y(s) \qquad \mathcal{L}(g(t)) = G(s)$$

$$\therefore (s^2+1)Y(s) = G(s)$$

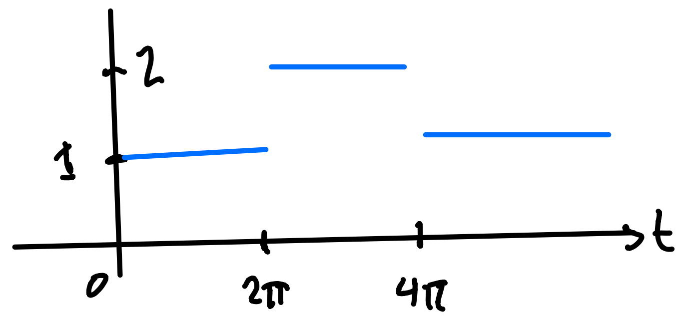
$$Y(s) = G(s)\frac{1}{s^2+1} \equiv G(s)F(s)$$

$$\therefore y(t) = g(t) * f(t) = \int_0^t g(\tau)\sin(t-\tau)\,d\tau$$

So now you have a general solution to IVP which you can evaluate explicitly by computing the convolution integral, if you are given $g(t)$

For example, suppose $g(t)=\begin{cases}1 & 0<t<2\pi \\ 2 & 2\pi\le t\le 4\pi \\ 1 & t\ge 4\pi\end{cases}$

$$g(t) = \mu_0 - \mu_{2\pi} + 2\mu_{2\pi} - 2\mu_{4\pi} + \mu_{4\pi}$$



$$g(t) = \mu_0 + \mu_{2\pi} - \mu_{4\pi}$$

$$y(t) = \int_0^t g(\tau)\sin(t-\tau)\,d\tau$$

$u = t-\tau \quad du = -d\tau$

$t \quad 0$

For $0 \le t < 2\pi$

$$y(t) = \int_0^t \sin(t-\tau)\,d\tau = -\int_t^0 \sin u\,du = \int_0^t \sin u\,du = -\cos t + 1$$

For $2\pi \le t \le 4\pi$

$$y(t) = -\cos t + 1 + \int_{2\pi}^t \sin(t-\tau)\,d\tau = \int_{2\pi}^t \sin u\,du = 2(-\cos t + 1)$$

$$y(t) = 2(-\cos t + 1) - \int_{4\pi}^{t} \sin(t-\tau)\,d\tau = -\cos t + 1$$

$$\therefore \quad y(t) = \begin{cases} 1-\cos t & 0 \le t < 2\pi \\ 2(1-\cos t) & 2\pi \le t < 4\pi \\ 1-\cos t & t > 4\pi \end{cases} \quad (\neq)$$

Compare this result with:

$$\begin{cases} y'' + y = g(t) = u_0(t) + u_{2\pi}(t) - u_{4\pi}(t) \\ y(0) = y'(0) = 0 \end{cases}$$

$$Y(s) = \frac{1}{s^2+1}\left[\frac{1}{s} + \frac{e^{-2\pi s}}{s} - \frac{e^{-4\pi s}}{s}\right]$$

$$Y(s) = \frac{1}{(s^2+1)s}\left[1 + e^{-2\pi s} - e^{4\pi s}\right]$$, Partial fractions:

$$\frac{1}{(s^2+1)s} = \frac{As+B}{s^2+1} + \frac{C}{s}$$

$$1 = As^2 + Bs + Cs^2 + C \Rightarrow C = 1,\ B = 0$$
$$A = -C = -1$$

$$\frac{1}{(s^2+1)s} = -\frac{s}{s^2+1} + \frac{1}{s}$$

$$-\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = -\cos t \qquad \mathcal{L}\left(\frac{1}{s}\right) = 1$$

$$\therefore\ y(t) = \mathcal{L}^{-1}(Y(s)) = -\cos t + 1 - \cos(t-2\pi)u_{2\pi} + u_{2\pi}$$
$$+\cos(t-4\pi)u_{4\pi} - u_{4\pi}$$
$$(-\cos t + 1)\left[1 + u_{2\pi} - u_{4\pi}\right]$$

$$y(t) = \begin{cases} 1-\cos t & 0 \le t < 2\pi \\ 2(1-\cos t) & 2\pi < t \le 4\pi \\ 1-\cos t & t \ge 4\pi \end{cases}$$

Compare to $(\ddagger)$ //

ANOTHER IMPORTANT APPLICATION:

SOLUTION OF LINEAR INTEGRAL EQUATIONS

The Volterra Integral Equation

is given by

$$(\ast) \qquad \varphi(t) + \int_0^t (t-\xi)\varphi(\xi)\,d\xi = 1$$

Find $\varphi(t)$

Notice that we have a convolution in the second term. Use Laplace Transforms:

$$\mathcal{L}(\varphi(t)) = \Phi(s)$$

$$\int_0^t (t-\xi)\varphi(\xi)\,d\xi = t * \varphi(t)$$

$$\mathcal{L}(t * \varphi(s)) = \frac{1}{s^2}\Phi(s)$$

$\therefore$ (*) after Laplace transforms, is

$$\Phi + \frac{1}{s^2}\Phi = \frac{1}{s}$$

$$\Rightarrow \quad \frac{s^2+1}{s^2}\Phi = \frac{1}{s}$$

or $\Phi = \dfrac{s}{s^2+1}$ $\quad \mathcal{L}^{-1}(\Phi(s)) = \cos t$ //

ex) $\begin{cases} \varphi'(t) + 2\varphi(t) + \int_0^t \varphi(\xi)\,d\xi = 1 & (1) \\ \varphi(0) = 0 \end{cases}$

Solve for $\varphi(t)$: Write (1) as

$$\varphi' + 2\varphi + 1 * \varphi = 1$$

Let $\Phi(s) = \mathcal{L}(\varphi(t))$

$$\mathcal{L}(\varphi') = s\Phi(s) - \varphi(0) = s\Phi(s)$$

$$\mathcal{L}(1 * \varphi(t)) = \frac{1}{s}\Phi(s)$$

$$\therefore\ s\Phi(s) + 2\Phi + \frac{1}{s}\Phi(s) = \frac{1}{s}$$

$$(s^2 + 2s + 1)\Phi = 1$$

$$\Phi = \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}(\Phi) = te^{-t}$$

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ex) Find solution to

$$\varphi(t) + 2\int_0^t \cos(t-\xi)\,\phi(\xi)\,d\xi = e^{-t}$$

let $\Phi(s) = \mathcal{L}(\varphi(t))$

so $\varphi(t) + 2\cos t * \varphi = e^{-t}$, after Laplace:

$$\Phi(s) + \frac{2s}{s^2+1}\Phi(s) = \frac{1}{s+1}$$

$$\Phi(s)\left(\frac{s^2+2s+1}{s^2+1}\right) = \frac{1}{s+1}$$

$$\Phi(s)\left(\frac{(s+1)^2}{s^2+1}\right) = \frac{1}{s+1}$$

$$\Phi(s) = \frac{s^2+1}{(s+1)^3} = \frac{s^2+2s+1-2s}{(s+1)^3} = \frac{1}{s+1} - \frac{2s}{(s+1)^3}$$

$$= \frac{1}{s+1} - 2\left[\frac{s+1}{(s+1)^3} - \frac{1}{(s+1)^3}\right]$$

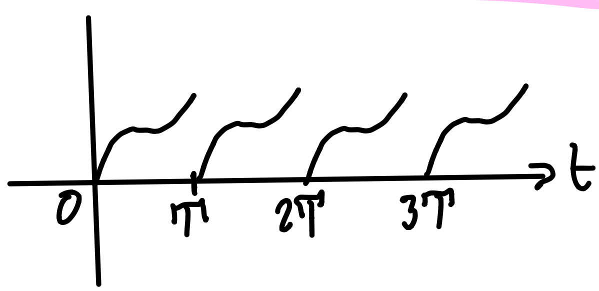
$$= \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

$\mathcal{L}^{-1}(F(s)) = e^{-t} - 2te^{-t} + t^2 e^{-t} = (1-t)^2 e^{-t}$ //

TRANSFORM OF A PERIODIC FUNCTION:

Suppose $f(t)$ is periodic, piecewise continuous on $[0,\infty)$, with period T

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$$



Since $\mathcal{L}(f(t)) = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$

let $t = u+T \quad dt = du$

$$\int_T^\infty = \int_0^\infty e^{-s(u+T)} f(u+T)\,du = e^{-sT}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-sT} F(s)$$

$$\int_0^T e^{-st} f(t)\,dt = (1 - e^{-sT}) \int_0^\infty e^{-st} f(t)\,dt = (1 - e^{-sT}) F(s)$$

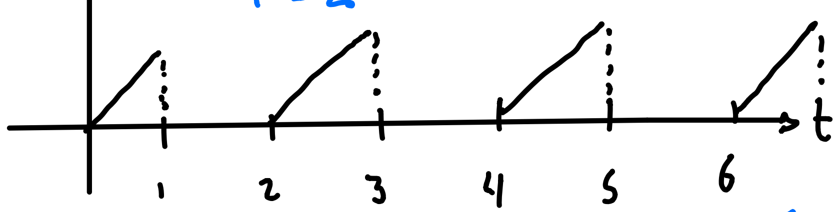
$$F(s) = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

ex) Find $\mathcal{L}(f(t))$ where

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

such that $f(t+2) = f(t)$ $\quad t \ge 0$

$T = 2$



Let $G(s) = \int_0^\infty e^{-st} t [u_0 - u_1]\,dt = \int_0^1 t e^{-st}\,dt$

IBP $\quad w = t \quad v = -\frac{1}{s} e^{-st}$

$\quad dw = dt \quad dv = e^{-st}\,dt$

$$G(s) = -\frac{1}{s}e^{-st}t\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt = -\frac{1}{s}e^{-s} + \frac{1}{s^2} - \frac{1}{s^2}e^{-s}$$

$$\therefore \mathcal{L}(f(t)) = \frac{1}{1-e^{-2s}}G(s) = \frac{1-(s+1)e^{-s}}{s^2(1-e^{-2s})} \quad //$$