6.5 Impulse Functions

We propose a function: "Generalized Functions" that depend on a parameter. They are not uniquely defined. A fundamental property is that their integral is equal to 1.

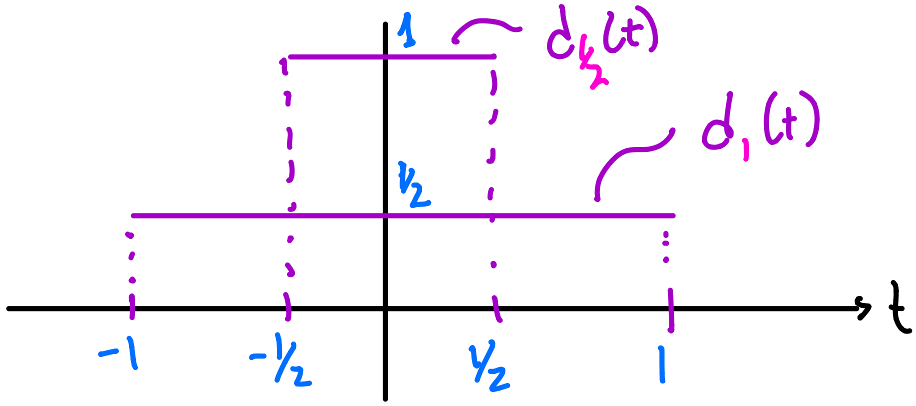
For example, we can propose a rectangular, generalized function:

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$

parameter

so that: $$\int_{-\infty}^{\infty} d_\tau(t)\,dt = \int_{-\tau}^{\tau} \frac{dt}{2\tau} = \frac{t}{2\tau}\Big|_{-\tau}^{\tau} = 1.$$

A picture of $d_\tau(t)$:



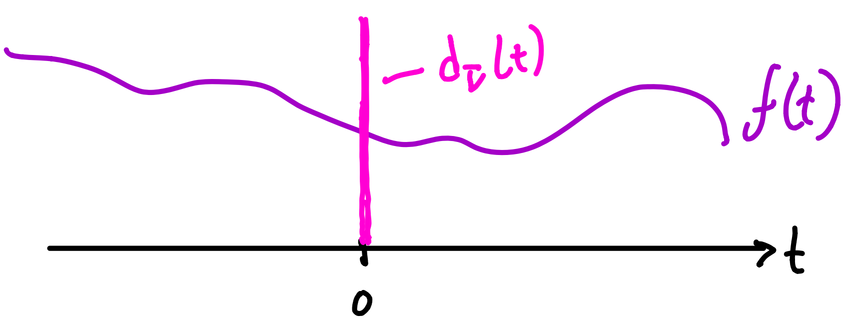
We add another constraint to this parametric function: it must be continuous at $t=0$.

The above $d_\tau(t)$ is certainly so.

With this added property,

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty} d_\tau(t)\, f(t)\, dt$$

$$= \lim_{\tau\to 0}\int_{-\tau}^{\tau}\frac{1}{2\tau} f(t)\, dt = f(0)$$



In fact, we don't have to center the generalized function at $t=0$, we could center it at $t=t_0$:

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0)\, f(t)\, dt = f(t_0)$$

THE SIFTING PROPERTY

In the $\lim_{\tau \to 0} d_\tau(t-t_0) \equiv \delta(t-t_0)$

The Unit Impulse or Dirac Delta Function

It's called a "function", but it is not really a function in the conventional sense. They are called Distributions.

THE DIRAC DELTA FUNCTION:

$$\delta(t-t_0) = \begin{cases} 0 & t \neq t_0 \\ \text{non-zero, otherwise} & \end{cases}$$

i.e. at $t=t_0$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

and obeys the sifting property

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$$

The Laplace transform of $\delta(t-t_0)$:

$$\mathcal{L}\left(\delta(t-t_0)\right) = \int_{0}^{\infty} \delta(t-t_0)\, e^{-st}\,dt$$

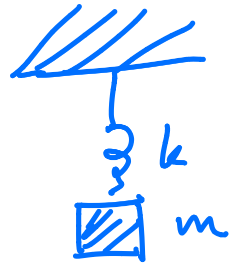
$t_0 \geq 0$

$$= e^{-st_0}$$

we used the sifting property

One important application of Dirac Delta functions is in modeling impulsive forces:

ex)



IVP $\begin{cases} y'' + y = \delta(t - 2\pi) \\ y(0) = y'(0) = 0 \end{cases}$

Take $\mathcal{L}$ of IVP. Let $Y(s) = \mathcal{L}(y(t))$,

so

$$\mathcal{L}(y'' + y) = \mathcal{L}(\delta(t - 2\pi))$$

$$s^2 Y(s) + Y(s) = e^{-2\pi s}$$

$$(s^2 + 1) Y(s) = e^{-2\pi s}$$

$$Y(s) = \frac{e^{-2\pi s}}{s^2 + 1} = F(s)\, e^{-2\pi s}$$

$$\mathcal{L}^{-1}(Y(s)) = u_{2\pi}(t) \sin(t - 2\pi) = u_{2\pi}(t) \sin t$$

$$y(t) = \begin{cases} 0 & 0 \le t < 2\pi \\ \sin t & t \ge 2\pi \end{cases}$$

ex) $y'' + 2y' + y = 3\delta(t-1)$ Damped, forced oscillator.

$\omega^2 = 1 = \frac{k}{m}$ spring const/mass

$\gamma = 2$ damping

$\omega = \sqrt{\frac{k}{m}}$ is "natural frequency"

$y(0) = 1$ $y'(0) = 1$ I.C.

Let $Y(s) = \mathcal{L}(y(t))$

$$\mathcal{L}(y'') = s^2 Y(s) - sy(0) - y'(0)$$
$$= s^2 Y(s) - s - 1$$

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(3\delta(t-1)) = 3e^{-s}$$

$$\therefore \quad s^2 Y(s) - s - 1 + 2sY(s) - 2 + Y(s) = 3e^{-s}$$

Solve for $Y(s)$:

$$(s^2+2s+1)Y(s)-s-3 = 3e^{-s}$$

$$(s+1)^2 Y(s) = 3e^{-s}+s+3$$

$$Y(s) = \frac{3e^{-s}+s+3}{(s+1)^2}$$

$$Y(s) = \frac{3e^{-s}}{(s+1)^2} + \frac{s+1}{(s+1)^2} + \frac{2}{(s+1)^2}$$

$$Y(s) = \frac{3e^{-s}}{(s+1)^2} + \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

From Tables:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t \qquad \mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$\mathcal{L}^{-1}\left(F(s+1)\right) = e^{-t}f(t)$$

$$\mathcal{L}^{-1}\left(e^{-cs}F(s)\right) = \mu_c(t)\,f(t-c)$$

$$\therefore y(t) = e^{-t} + 2te^{-t} + 3\mu_1(t)(t-1)e^{-(t-1)}$$

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ex) $y'' + 2y' + 2y = \sum_{l=1}^{3} k\,\delta(t - l\pi)$

$$= k\left[\delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi)\right]$$

$y(0) = 0 \qquad y'(0) = 1$

let $Y(s) = \mathcal{L}(y(t))$

Take $\mathcal{L}$ (IVP)

$$s^2 Y(s) - 1 + 2s Y(s) + 2Y(s)$$

$$= k \sum_{l=1}^{3} e^{-l\pi s}$$

$$(s^2 + 2s + 2) Y(s) - 1 = k \sum_{l=1}^{3} e^{-l\pi s}$$

$$Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{k}{s^2 + 2s + 2} \sum_{l=1}^{3} e^{-l\pi s}$$

$$Y(s) = \frac{1}{(s+1)^2 + 1} + \frac{k}{(s+1)^2 + 1} \sum_{l=1}^{3} e^{-l\pi s}$$

From Tables: $\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$

$\mathcal{L}^{-1}\left(F(s-a)\right) = e^{at} f(t)$

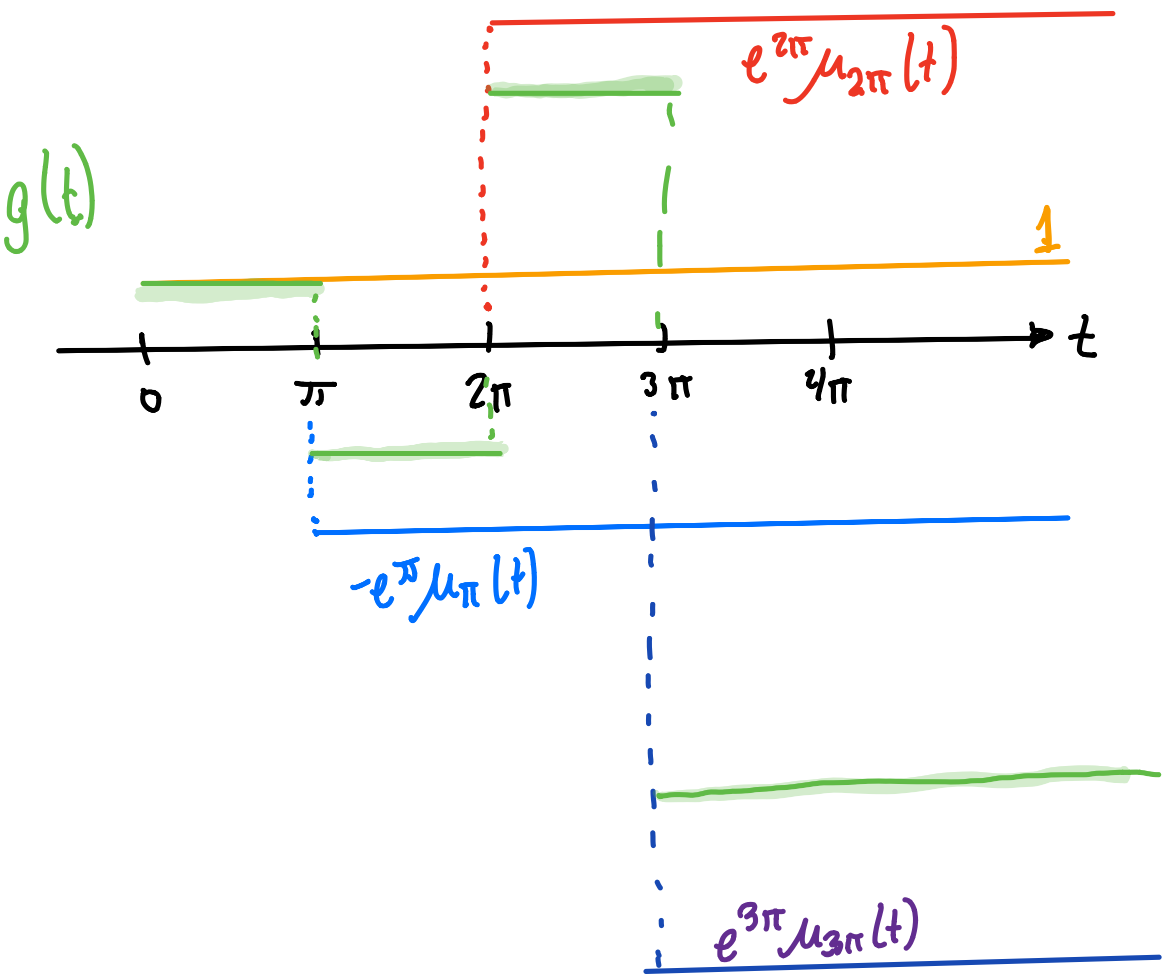
$$y(t) = \sin t\, e^{-t} + \underline{k\, u_{\pi}(t)\, e^{-(t-\pi)} \sin(t-\pi)}$$
$$+ \underline{k\, u_{2\pi}(t)\, e^{-(t-2\pi)} \sin(t-2\pi)}$$
$$+ \underline{k\, u_{3\pi}(t)\, e^{-(t-3\pi)} \sin(t-3\pi)}$$

$$y(t) = \sin t\, e^{-t} - k\, u_{\pi}(t)\, e^{-(t-\pi)} \sin t$$
$$+ k\, u_{2\pi}(t)\, e^{-(t-2\pi)} \sin t$$
$$- k\, u_{3\pi}(t)\, e^{-(t-3\pi)} \sin t$$

$$= \sin t\, e^{-t}\left[1 - k\, u_{\pi}(t)\, e^{\pi} + k\, u_{2\pi}(t)\, e^{2\pi} - k\, u_{3\pi}(t)\, e^{3\pi}\right]$$

$$y(t) = \sin t\, e^{-t}\left[1 + k \sum_{l=1}^{3} (-1)^{l} u_{l\pi}(t)\, e^{l\pi}\right]$$

Let $k=1$ $\quad g(t) = 1 - e^{\pi}u_{\pi}(t) + e^{2\pi}u_{2\pi}(t) - e^{3\pi}u_{3\pi}(t)$



$y(t) = \sin t \, e^{-t} g(t)$ with $k=1$