SECTION 6.4 Initial Value Problems with discontinuous forcing

By forcing we mean the non-homogeneous function $f(t)$ in

$$Ly(t) = f(t)$$

Consider

IVP $\begin{cases} y'' + y = f(t) & t > 0 \quad \textbf{ODE} \\ y(0) = y'(0) = 0 & \quad \textbf{I.C.} \end{cases}$

with $f(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$

$\therefore \; f(t) = 1 - u_\pi(t)$

$$y'' + y = 1 - u_\pi(t) \qquad t > 0$$

Take $\mathcal{L}$(ODE) & use I.C.

let $Y(s) = \mathcal{L}(y(t))$

$$\mathcal{L}(y'') = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s)$$

$$\therefore \quad (s^2+1)\,Y(s) = \mathcal{L}\left(1 - u_\pi(t)\right) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$\therefore \; Y(s) = \frac{1}{(s^2+1)s}\left[1 - e^{-\pi s}\right]$$

$$= \left[\frac{1}{s} - \frac{s}{s^2+1}\right]\left[1 - e^{-\pi s}\right]$$

$\frac{1}{(s^2+1)s} = \frac{1}{s} - \frac{s}{s^2+1}$ follows by inspection.

Alternatively, found using Partial fractions:

$$\frac{1}{(s^2+1)s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Multiply b.s. by $(s^2+1)s$

$$1 = A(s^2+1) + Bs^2 + Cs = As^2 + A + Bs^2 + Cs$$

$\therefore \quad A = 1$

$A + B = 0 \Rightarrow B = -1$

$C = 0$

$$\therefore \frac{1}{(s^2+1)s} = \frac{1}{s} - \frac{s}{s^2+1} \quad //$$

So $\quad Y(s) = \left[\frac{1}{s} + \frac{-s}{s^2+1}\right](1 - e^{-\pi s}) = G(s)(1 - e^{-\pi s})$

$$\mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) = 1 - \cos t$$

$$\therefore \mathcal{L}^{-1}(Y(s)) = 1 - \cos t - u_\pi(t)\left[1 - \cos(t-\pi)\right] = y(t)$$

$$\therefore y(t) = \begin{cases} 1 - \cos t & 0 \le t < \pi \\ 1 - \cos t - u_\pi\left[1 - \cos(t-\pi)\right] \end{cases}$$

since $\cos(t-\pi) = -\cos t$

$$y(t) = \begin{cases} 1-\cos t & 0 \le t < \pi \\ -2\cos t & t \ge \pi \end{cases}$$