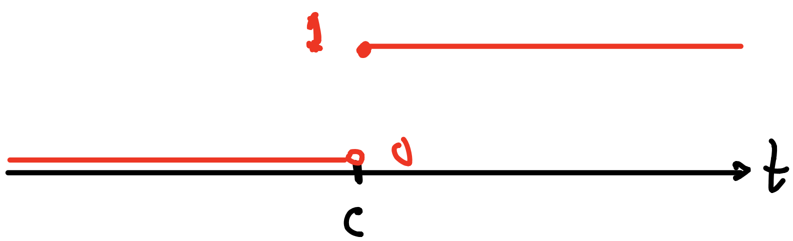
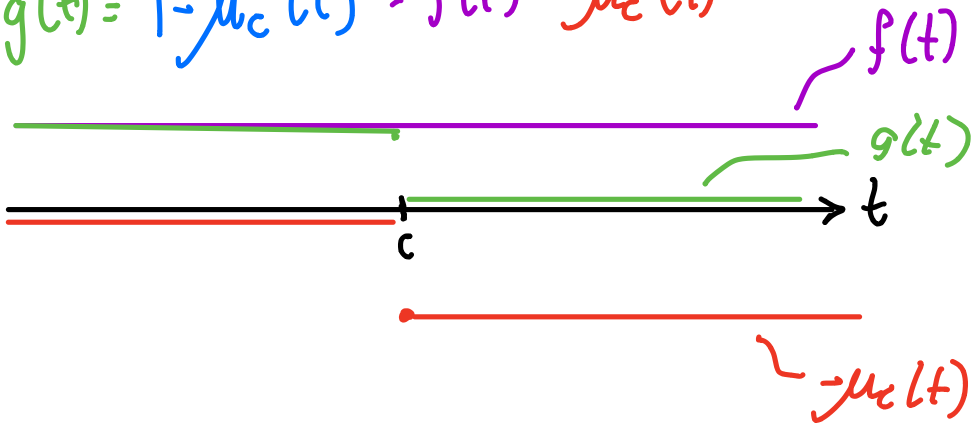
6.3 THE STEP FUNCTION (HEAVISIDE FUNCTION)

The Heaviside Function

$$u_c(t) \equiv \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



$$g(t) = 1 - u_c(t) = f(t) - u_c(t)$$

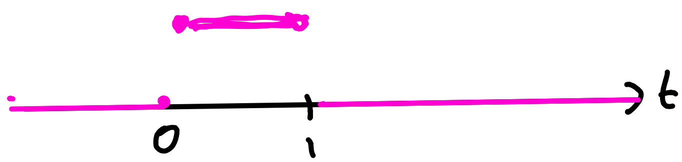


ex) Represent $f(t) = \begin{cases} 0 & 0 \leq t < 3 \\ 1 & 3 \leq t < 5 \\ 0 & t > 5 \end{cases}$

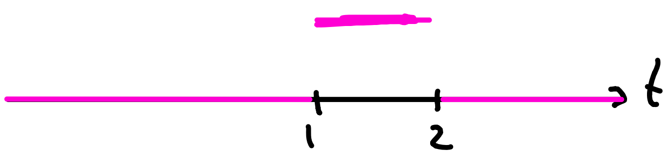
$$f(t) = \mu_3(t) - \mu_5(t)$$

ex) $$g(t) = \begin{cases} t & 0 \le t < 1 \\ t^2 & 1 < t \le 2 \\ t^3 & t > 2 \end{cases}$$

Can be written in terms of step functions:



A $\mu_0(t) - \mu_1(t)$



B $\mu_1(t) - \mu_2(t)$

C $\mu_2(t)$

Using A - C:

$$g(t) = \big(\mu_0(t) - \mu_1(t)\big)\, t + \big(\mu_1(t) - \mu_2(t)\big)\, t^2 + \mu_2(t)\, t^3$$

The Translation Theorem:

If $F(s) = \mathcal{L}(f(t))$ exists, for $s > a \geq 0$ and $c > 0$

then

(†) $$\mathcal{L}\left(u_c(t)\, f(t-c)\right) = e^{-cs} F(s).$$

This can be verified:

$$\mathcal{L}\left(u_c(t) f(t-c)\right) = \int_0^\infty u_c(t)\, f(t-c)\, e^{-st}\, dt$$

since $u_c(t) = 0$ for $t < c$

$$= \int_c^\infty f(t-c)\, e^{-st}\, dt$$

let $u = t - c$ $\quad t = u + c$

$du = dt$

$$= \int_0^\infty f(u)\, e^{-(u+c)s}\, du = e^{-cs} \int_0^\infty f(u)\, e^{-us}\, du = e^{-cs} F(s)$$

//

ANOTHER TRANSLATION THEOREM:

(✗) $$\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c))$$

Let's verify: $\mathcal{L}(u_c(t)f(t)) = \int_0^\infty u_c(t)f(t)e^{-st}dt$

$$= \int_c^\infty f(t)e^{-st}dt \qquad \text{let } u = t-c \quad \therefore t = u+c$$

$$= \int_0^\infty f(u+c)e^{-s(u+c)}du = e^{-cs}\int_0^\infty f(u+c)e^{-su}du$$

$$= e^{-cs}\mathcal{L}(f(t+c)) \quad //$$

<u>Rule</u>: Might have difficulties finding $\mathcal{L}(f(t+c))$ in a Table, but sometimes one can manipulate $f(t+c)$ to yield a familiar Laplace transform.

Let use the above 2 translation theorems to find transforms.

(we also want to make a point that we tell the difference between these)

ex) Find $\mathcal{L}^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right) = \mathcal{L}^{-1}\left(e^{-\pi s}F(s)\right)$

$$\mathcal{L}^{-1}\left(F(s)\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$\therefore$ using $(\ddagger)$ $\mathcal{L}^{-1}\left(e^{-\pi s}F(s)\right) = u_\pi(t)\sin(t-\pi)$

$$= -u_\pi(t)\sin t$$

since $\sin(t-\pi) = -\sin t$

ex) Find $\mathcal{L}(g(t))$ where $g(t) = \begin{cases} 0 & 0 \le t < \frac{\pi}{4} \\ \sin t & t > \frac{\pi}{4} \end{cases}$

$$g(t) = \sin t \; u_{\frac{\pi}{4}}(t)$$

Note: cannot use (‡). Could use (¥) but require some algebra to find inverses in table:

$$\mathcal{L}\left(\sin t \, u_{\frac{\pi}{4}}(t)\right) = e^{-\frac{\pi s}{4}} \mathcal{L}\left(\sin\left(t+\frac{\pi}{4}\right)\right)$$

using (¥). The problem is that $\mathcal{L}\left(\sin\left(t+\frac{\pi}{4}\right)\right)$ is not in the Table. Use Euler's formula $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\mathcal{L}\left[\frac{1}{2i} e^{i\left(t+\frac{\pi}{4}\right)} - \frac{1}{2i} e^{-i\left(t+\frac{\pi}{4}\right)}\right]$$

$$= \frac{1}{2i} e^{i\pi/4} \mathcal{L}(e^{it}) - \frac{1}{2i} e^{-i\pi/4} \mathcal{L}(e^{-it})$$

$$= \frac{1}{2i} e^{i\pi/4} \frac{1}{s-i} - \frac{1}{2i} e^{-i\pi/4} \frac{1}{s+i}$$

using $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ (Table)

Now it's a matter of making the result tidier:

$$e^{\pm i\pi/4} = \cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4} = \sqrt{2} \pm i\sqrt{2}$$

$$= \frac{1}{2i}\left[\frac{1}{s-i}\left(\sqrt{2}+i\sqrt{2}\right) - \frac{1}{s+i}\left(\sqrt{2}-i\sqrt{2}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{s+i}{s^2+1}\left(\sqrt{2}+i\sqrt{2}\right) - \frac{s-i}{s^2+1}\left(\sqrt{2}-i\sqrt{2}\right)\right]$$

$$= \frac{1}{2i}\frac{\sqrt{2}}{s^2+1}\left[(s+i)(1+i) - (s-i)(1-i)\right]$$

$$\begin{array}{r} s+i+is-1 \\ -(s-i-is-1) \\ \hline 2i+2is \end{array}$$

$$= \frac{1}{2i}\frac{\sqrt{2}}{s^2+1}\left[\cancel{s}+i+is-1 - (\cancel{s}-i-is-1)\right]$$

$$= \frac{1}{2i}\frac{\sqrt{2}}{s^2+1}\left[2i+2is\right] = \frac{2i}{2i}\frac{\sqrt{2}}{s^2+1}\left[1+s\right]$$

$$= \frac{\sqrt{2}}{s^2+1} + \frac{\sqrt{2}\,s}{s^2+1}$$

$$\therefore \mathcal{L}\left(\sin\left(t+\frac{\pi}{4}\right)\right) = \sqrt{2}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

Finally

$$\mathcal{L}\left(u_{\frac{\pi}{4}}(t)\sin t\right) = \sqrt{2}\, e^{-\frac{\pi}{4}s}\left(\frac{1+s}{s^2+1}\right)$$

ex) $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^3}\right)$

know that $\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = t^2$

Using $(\ddagger)$:

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^3}\right) = \frac{1}{2} u_3(t)\,(t-3)^2$$

ex) $\mathcal{L}\left(u_2(t)\,t^3\right) = \mathcal{L}\left((t+2)^3\right)e^{-2s}$

$$= e^{-2s}\,\mathcal{L}\left(t^3 + 6t^2 + 12t + 8\right)$$

$$= e^{-2s}\left[\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}\right]$$

ANOTHER TRANSLATION THEOREM

If $F(s) = \mathcal{L}(f(t))$ then

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

$s > a \geq 0$

Verify:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{at} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(s-a)t} dt$$

$$= F(s-a)$$

ex) $\mathcal{L}(e^{3t}\sin t) = \dfrac{1}{(s-3)^2+1}$

since $\mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$

ex) $\mathcal{L}(t^2 e^{3t}) = \dfrac{2}{(s-3)^3}$

since $\mathcal{L}(t^2) = \dfrac{2}{s^3}$

//