6.2 Inverse Laplace Transforms (by Table Lookup)

You'll find Tables of Laplace Transforms. We use these:

ex) Find $\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\,\frac{1}{2}\right)$

In Table we find $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}\sin 2t \quad //$$

ex) Find $\mathcal{L}^{-1}\left(\frac{s-1}{s^2-s-6}\right)$. Use partial fractions:

$$\frac{1}{s^2-s-6} = \frac{1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$1 = A(s-3) + B(s+2) \Rightarrow A+B=0$$

$$\therefore A = -B$$

$$-3A+2B=1 \quad \therefore B = \frac{1}{5} \quad \therefore A = -\frac{1}{5}$$

hence

$$\frac{s-1}{(s-3)(s+2)} = -\frac{1}{5}\,\frac{s-1}{s+2} + \frac{1}{5}\,\frac{s-1}{s-3}$$

dividing:

$$= \frac{3}{5}\,\frac{1}{s+2} + \frac{2}{5}\,\frac{1}{s-3}$$

In table $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$$\mathcal{L}^{-1}\left(\frac{s-1}{s^2-s-6}\right) = \frac{3}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \frac{2}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{3}{5}e^{-2t} + \frac{2}{3}e^{3t} \quad //$$

ex) $\mathcal{L}^{-1}\left(\frac{s}{s^2-4s+13}\right)$ completing the square:

$$s^2-4s+13 = s^2-4s+4-4+13$$

$$= (s-2)^2+9 \quad \therefore$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s-2)^2+9}\right) = \mathcal{L}^{-1}\left(\frac{s-2+2}{(s-2)^2+3^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + \mathcal{L}^{-1}\left(\frac{3\cdot 2/3}{(s-2)^2+3^2}\right)$$

In table $\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

In table $\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin bt$

$$\therefore \quad \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) = e^{2t}\cos 3t$$

$$\frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{(s-2)^2+3^2}\right) = \frac{2}{3}e^{2t}\sin 3t$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s}{s^2-4s+13}\right) = e^{2t}\cos 3t + \frac{2}{3}e^{2t}\sin 3t$$

Laplace Transform of the derivative of $f(t)$:

$$\mathcal{L}(f'(t)) = \int_0^\infty \frac{df}{dt} e^{-st} dt$$

IBP

let $u = e^{-st}$ $\quad v = f$

$du = -se^{-st} dt \quad dv = \frac{df}{dt} dt$

$$\int_0^\infty f' e^{-st} dt = f e^{-st} \Big|_0^\infty + s \int_0^\infty f e^{-st} dt$$

$$- f(0) \quad + s F(s)$$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t)) = \mathcal{L}\left(\frac{d}{dt}f'\right) = \mathcal{L}(g'(t))$$

$$= s\,G(s) - g(0)$$

$$= s^2 F(s) - s\,f(0) - f'(0)$$

by induction

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{(n-1)}(0)$$

We can use this to solve IVP:

Let $Y(s) = \mathcal{L}(y(t))$

ex)

$$\text{IVP}\begin{cases} y'' - y = e^t \\ y(0) = 1 \quad y'(0) = 0 \end{cases}$$

① $\mathcal{L}$ Transform ODE, use of I.C. $\Rightarrow Y(s)$

② $\mathcal{L}^{-1}(Y(s)) = y(t)$

$$\mathcal{L}(y''-y) = \mathcal{L}(e^t) \quad (\ddagger)$$

$$\mathcal{L}(y'') = s^2 Y(s) - s y(0) - y'(0)$$

$$= s^2 Y(s) - s$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(e^t) = \frac{1}{s-1} \quad \text{(by Table)}$$

$$\therefore (\ddagger) \quad s^2 Y(s) - s - Y(s) = \frac{1}{s-1}$$

$$(s^2-1)Y(s) = \frac{1}{s-1} + s =$$

$$Y(s) = \frac{1}{s^2-1}\left[\frac{1}{s-1} + s\right]$$

$$Y(s) = \frac{1}{s+1} + \frac{s}{(s+1)(s-1)^2}$$

Step 2:

$$\mathcal{L}^{-1}(Y(s)) = \underbrace{\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)}_{\mathrm{I}} + \underbrace{\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s-1)^2}\right)}_{\mathrm{II}}$$

$$y(t) = \mathrm{I} + \mathrm{II} = e^{-t} + \mathrm{II}$$

//

$$\mathrm{II}: \frac{s}{(s+1)(s-1)^2} = s\left[\frac{1}{(s+1)(s-1)^2}\right]$$

$$= s\left[\frac{A}{s+1} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}\right]$$

//

$$1 = \left[\frac{A}{s+1} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}\right]$$

$$1 = A(s-1)^2 + B(s+1)(s-1) + C(s+1)$$

$$1 = A(s^2-2s+1) + B(s^2-1) + Cs + C$$

$$A = -B$$

$$-2A + C = 0 \Rightarrow C = 2A = -2B$$

$$1 = A - B + C = -B - B - 2B = -4B$$

$$B = -\frac{1}{4} \quad A = \frac{1}{4} \quad C = \frac{1}{2}$$

$$\text{II}: s\left[\frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\right]$$

$$= \frac{1}{4}\frac{s}{s+1} - \frac{1}{4}\frac{s}{s-1} + \frac{1}{2}\frac{s}{(s-1)^2}$$

from table

$$\mathcal{L}^{-1}\left(\frac{s}{s+1}\right) = \cos t$$

$$\mathcal{L}^{-1}\left(\frac{s}{s-1}\right) = \cosh t$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s-1)^2}\right) = \mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-1} + \frac{1}{(s-1)^2}\right)$$

let $c = 1$

In Table: $e^{ct} f(t) \xrightarrow{\mathcal{L}} F(s-c)$

$$\mathcal{L}^{-1}(F(s)) = 1 = f(t)$$

$\therefore$

II: $\frac{1}{4}\cos t - \frac{1}{4}\cosh t + \frac{1}{2} e^{t}$

$$\therefore y(t) = \underbrace{e^{-t}}_{\text{I}} + \underbrace{\frac{1}{4}\cos t - \frac{1}{4}\cosh t + \frac{1}{2} e^{t}}_{\text{II}} \quad //$$

ex) Using Laplace transforms of derivatives to find a Laplace transform of another function

Find $\mathcal{L}(t\cos at) = \int_0^\infty t\cos at\, e^{-st}\,dt$

can be done by IBP after using

$\cos at = \frac{1}{2}(e^{iat} + e^{-iat})$

Smaller Way: if $F(s) = \int_0^\infty f(t)\, e^{-st}\,dt$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\,dt = \int_0^\infty (-t)\, f(t)\, e^{-st}\,dt$$

$$\therefore \quad \frac{dF}{ds} = -\mathcal{L}(t f(t)) \quad (\star)$$

In general

$$\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$$

$\mathcal{L}(t\cos at)$ let $f(t) = \cos at$

Apply (☆)

from table $\mathcal{L}(\cos at) = F(s) = \dfrac{s}{s^2+a^2}$

$$\mathcal{L}(t\cos at) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

using $\left(\dfrac{u}{v}\right)' = \dfrac{udv - vdu}{v^2}$

i.e. $-\dfrac{d}{ds}\left(\dfrac{s}{s^2+a^2}\right) = -\dfrac{s(2s)-(s^2+a^2)}{(s^2+a^2)^2} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$