6.1 Laplace Transforms

Useful for finding solutions to n^{th} order linear IVP. We'll use

$\mathcal{L}$ to indicate Laplace Transform, and
$\mathcal{L}^{-1}$ to indicate an inverse laplace transform.

e.g. $y(t)$ given, then

$$\mathcal{L}\, y(t) = Y(s)$$

$$\mathcal{L}^{-1}\, Y(s) = y(t)$$

The Laplace transform applies to any piecewise continuous function $f(t)$, for $t \geq 0$ such that $|f(t)| \leq Ke^{at}$, K constant, for some $t \geq M \geq 0$. The a is any real number //

The Laplace Transform of $f(t)$:

$$\mathcal{L}(f(t)) \equiv F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

There are other transforms: Fourier, Hankel, Radon, z-Transform, etc. They are all of the form

$$\int_\alpha^\beta K(s,t)\, f(t)\, dt = F(s)$$

K is the kernel ($K = e^{-st}$ in Laplace transform)

All transforms are LINEAR:

if $F(s) = \int_\alpha^\beta K f\, dt$ $\quad G(s) = \int_\alpha^\beta K g\, dt$

then $\int_\alpha^\beta k(f+g)\,dt = F(s) + G(s)$. Also,

if c is a constant, then

$$\mathcal{L}(c f(t)) = c F(s), \text{ i.e.}$$

$$c\int_\alpha^\beta k f \, dt = \int_\alpha^\beta c k f \, dt \quad //$$

To apply Laplace Transforms to solving IVP for $y(t)$: Transform ODE and use I.C. to obtain algebraic equations for $Y(s)$.

Solve for $Y(s)$, then take inverse Laplace transform.

Rmk: the forward $t \to s$ via Laplace transforms is easy. What may be hard is finding the inverse of $Y(s)$: $\mathcal{L}^{-1}(Y(s)) = y(t)$.

Why? if $\mathcal{L}(y(t)) = Y(s)$, then the complex integral, if solvable, gives the inverse:

$$y(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} Y(s)e^{st}\,ds$$

You'll learn how to solve this type of integral in a complex variables class.

We will find inverses by using algebra, properties of transforms, and familiar transforms to find inverse transforms (see sections 6.2 - 6.5)

SOME SIMPLE EXAMPLES:

ex) Find $\mathcal{L}(1) = \int_0^\infty e^{-st} \, 1 \, dt$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

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ex) Find $\mathcal{L}(e^{at})$ a is a number

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} e^{-st} \, dt$$

$$= \int_0^\infty e^{(a-s)t} \, dt$$

$$= \frac{1}{(a-s)} e^{(a-s)t} \Big|_0^\infty = \frac{1}{s-a}, \quad s > a.$$

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ex) $\mathcal{L}(\cos\omega t) = \mathcal{L}\left(\frac{1}{2}e^{i\omega t} + \frac{1}{2}e^{-i\omega t}\right)$

$$= \frac{1}{2}\int_0^\infty e^{i\omega t}e^{-st}dt + \frac{1}{2}\int_0^\infty e^{-i\omega t}e^{-st}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-i\omega)t}dt + \frac{1}{2}\int_0^\infty e^{-(s+i\omega)t}dt$$

$$= \frac{1}{2}\frac{1}{s-i\omega} + \frac{1}{2}\frac{1}{s+i\omega} =$$

$$\frac{1}{2}\left[\frac{s+i\omega + s-i\omega}{s^2+\omega^2}\right] = \frac{s}{s^2+\omega^2} \quad //$$

ex) $\mathcal{L}(t) = \int_0^\infty te^{-st}dt = \lim_{R\to\infty}\left(\cancelto{0}{-\frac{t}{s}e^{-st}\Big|_0^R} + \frac{1}{s}\int_0^R e^{-st}dt\right)$

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$$= \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s^2}$$

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ex) $\mathcal{L}(t^p)$

We'll make use of the Gamma Function

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx$$

if p is an integer ≥ 0

$$\Gamma(p+1) = p!$$

$$\mathcal{L}(t^p) = \int_0^\infty t^p e^{-st}\,dt$$

Let $x = st$ $\quad t = x/s$

$dx = s\,dt$ $\quad dt = \frac{dx}{s}$

$$= \int_0^\infty \left(\frac{x}{s}\right)^p e^{-x}\,\frac{dx}{s} = \frac{1}{s^{p+1}}\Gamma(p+1)$$

if p is ≥ 0 integer

then $\frac{1}{s^{p+1}}\,p! = \mathcal{L}(t^p)$

$p = 0, 1, 2 \ldots, s > 0$

Rmk: How do we see that $\int_0^\infty x^p e^{-x}\,dx = p!$?

set $p=0$ $\quad \int_0^\infty x^0 e^{-x}\,dx = 1$

$p \geq 1$ $\quad \int_0^\infty x^p e^{-x}\,dx = -x^p e^{-x}\Big|_0^\infty + p\int_0^\infty x^{p-1} e^{-x}\,dx$

$$= p\,\Gamma(p)$$

$$\therefore \quad \Gamma(p+1) = p\,\Gamma(p)$$

with $(p=0)$ $\Gamma(1) = 1$

for example let $p=4$: $\Gamma(5) = 4\,\Gamma(4)$

$$= 4\cdot 3\,\Gamma(3)$$

$$= 4\cdot 3\cdot 2\cdot \Gamma(2)$$

$$= 4\cdot 3\cdot 2\cdot 1\cdot \Gamma(1) = 4!$$

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