3.8 Forced Oscillations

Consider

$$y'' + \beta y' + \omega^2 y = f(t)$$

rewrite this as

$$\text{IVP}\begin{cases} y'' + 2\lambda y + \omega^2 y = f(t) \\ \lambda = \dfrac{\gamma}{2m} \quad \omega^2 = \dfrac{k}{m} \quad f(t) = \dfrac{F(t)}{m} \\ \text{I.C.:} \quad y(0) = Y_0 \quad y'(0) = Y_1 \end{cases}$$

The general solution of the IVP:

$$y(t) = y_H(t) + y_p(t), \text{ where}$$

$$y_H'' + 2\lambda y_N' + \omega^2 y_H = 0$$

$$y_H = c_1 y_1 + c_2 y_2 \quad \text{and}$$

$$y_p'' + 2\lambda y_p' + \omega^2 y_p = f(t)$$

y_p found via MUC or variation of parameters.

ex) Let's think of ω as the driving frequency in the forcing $f(t)$:

$$y'' + 2\lambda y' + \omega_0^2 y = \sin \omega t$$

Start "at rest": $y(0) = y'(0) = 0.$

$$y_H = c_1 y_1 + c_2 y_2$$

$$y_H = e^{mt} \therefore$$

$$m^2 + 2\lambda m + \omega_0^2 = 0$$

$$m_{1,2} = -\lambda \pm \frac{1}{2}\sqrt{4\lambda^2 - 4\omega_0^2}$$

or $m_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$

Let's assume $\lambda = 1$ $\omega_0^2 = 3$, in what follows:

$$y_H(t) = e^{-t}\left[A\cos\sqrt{2}t + B\sin\sqrt{2}t\right]$$

Solve for y_p:

$$y_p'' + 2y_p' + 3y_p = \sin\omega t \quad (D)$$

Let $y_p = C_1\cos\omega t + C_2\sin\omega t \quad (A)$

$$\omega \neq \omega_0$$

$$y_p' = -C_1\omega\sin\omega t + C_2\omega\cos\omega t \quad (B)$$

$$y_p'' = -C_1\omega^2\cos\omega t - C_2\omega^2\sin\omega t$$

$$\therefore\ y_p'' = -\omega^2 y_p \quad (C)$$

substitute (A) - (C) into (D)

$$-\omega^2 y_p + 2(-C_1\omega\sin\omega t + C_2\omega\cos\omega t)$$

$$+ 3y_p = \sin\omega t$$

$$(3-\omega^2)(c_1 \cos\omega t + c_2 \sin\omega t)$$
$$- 2c_1 \omega \sin\omega t + 2c_2 \omega \cos\omega t$$
$$= \sin\omega t$$

Match terms proportional to $\cos\omega t$ & $\sin\omega t$:

$$\therefore \quad (3-\omega^2)\, c_1 + 2 c_2 \omega = 0 \qquad : \cos\omega t \text{ terms}$$
$$(3-\omega^2)\, c_2 - 2c_1 \omega = 1 \qquad : \sin\omega t \text{ terms}$$

$$\therefore \quad c_1 = \frac{-2\omega c_2}{(3-\omega^2)}$$

$$(3-\omega^2)c_2 + \frac{4\omega^2 c_2}{3-\omega^2} = 1 \text{ or } \frac{(3-\omega^2)^2 + 4\omega^2}{3-\omega^2} c_2 = 1$$

$$\begin{cases} c_2 = \dfrac{3-\omega^2}{(3-\omega^2)^2 + 4\omega^2} & (*) \\[2ex] c_1 = \dfrac{-2\omega}{(3-\omega^2)^2 + 4\omega^2} & (**) \end{cases}$$

$$\therefore \quad y = \left[A \cos\sqrt{2}t + B \sin\sqrt{2}t\right] e^{-t}$$
$$+ c_1 \cos\omega t + c_2 \sin\omega t \qquad (\dagger)$$

Apply I.C. on $(\ddagger)$. We'll need $y'(t)$:

$$y'(t) = -\left[A\cos\sqrt{2}t + B\sin\sqrt{2}t\right]e^{-t} + \left[-\sqrt{2}A\sin\sqrt{2}t + \sqrt{2}B\cos\sqrt{2}t\right]e^{-t} - \omega c_1\sin\omega t + \omega c_2\cos\omega t$$

Since $y(0)=0$, from $(\ddagger)$

$$0 = A + c_1 \Rightarrow A = -c_1 \quad (\text{see } *)$$

$$y'(0) = 0 = -A + \sqrt{2}B + \omega c_2$$

$$\Rightarrow B = \frac{A - \omega c_2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(c_1 + \omega c_2)$$

DISCUSSION:

Take the limit of $y(t)$ as $t \to \infty$ in $(\ddagger)$:

since $\lambda = 1$ the damping λ makes $y_H \to 0$ as $t \to \infty$. Hence,

$$\lim_{t\to\infty} y(t) = c_1\cos\omega t + c_2\sin\omega t = y_P = y_\infty$$

so $\left[A\cos\sqrt{2}t + B\sin\sqrt{2}t\right]e^{-t} = y_H = y_T$

def: the "transient part" y_T of the solution $y(t)$ in (‡) is the part that decays to 0 as $t \to \infty$. In this case, the y_T happens to be y_H.

def: the "asympotic part" y_∞ of the solution $y(t)$ in (‡) is the part of $y(t)$ that persists as $t \to \infty$. In this case y_P.

Note: if $\gamma = 0$ there is no transient solution. The solution is $y(t) = y_H + y_P$ always.

A useful identity: a solution of the form

$$c_1 \cos \omega t + c_2 \sin \omega t = R \cos(\omega t + \phi)$$

where $R = \sqrt{c_1^2 + c_2^2}$

$$\tan \phi = -\frac{c_2}{c_1}$$

RESONANCE: Consider again

$$y'' + 2\lambda y' + \omega_0^2 y = f_0 \cos \omega t$$

ω_0 is the "natural frequency of the spring/mass system. ω is the driving frequency. f_0 is the driving amplitude. Set f_0 constant, but now consider what happens when ω is varied:

in particular, let's examine what happens when

$\omega \sim \omega_0$

Again, the solution has a homogeneous part

$$y_H = e^{-\lambda t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$$

where $\omega_d = \sqrt{\omega_0^2 - \lambda}$. Here, $\omega_0^2 > \lambda$, the underdamped case.

In the limit, as $t \to \infty$, $y_H = 0$ (if $\lambda \neq 0$).

if $\omega_0^2 = \lambda$ (the critically damped case)

$$y_H = e^{-\lambda t}\left[A + Bt\right]$$

in this case $\lim_{t\to\infty} y_H = 0$ as well.

So in either the critically or underdamped cases, if $t \to \infty$

$$y \sim y_P = R\cos(\omega t + \phi)$$

$$R = \frac{f_0}{\Delta} \quad \text{where}$$

$$\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

Consider what happens to R as ω is varied:

if $\omega = \omega_0$ ("tuned") then

$$R = \frac{f_0}{\gamma\omega} \quad \text{if} \quad \gamma \neq 0.$$

if $\omega = \omega_0$ ("tuned"), but $\gamma = 0$ then

$\Delta = 0 \therefore R \to \infty$ (blows up!)