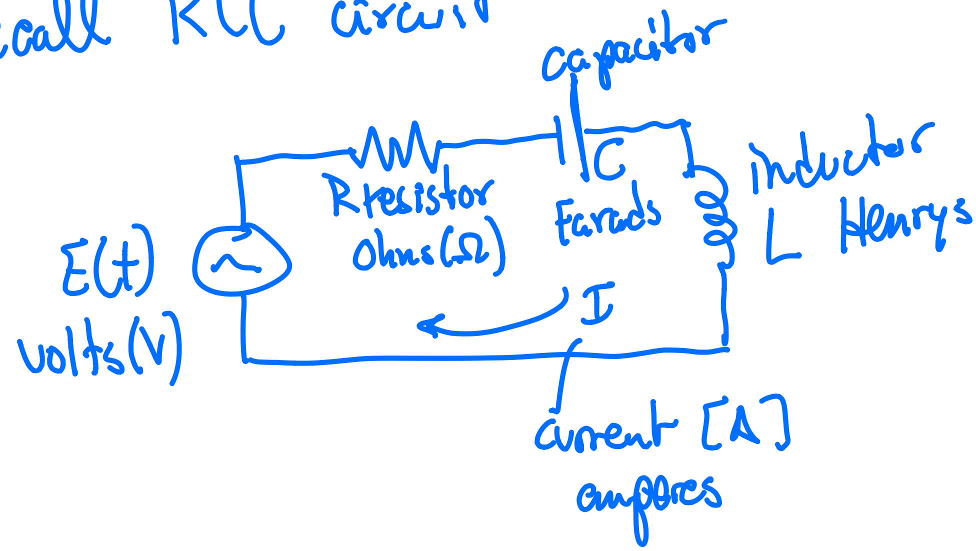
3.7 MECHANICAL & ELECTRICAL OSCILLATIONS

Recall RLC circuit



$$V_R = IR$$ voltage drop in R

$$V_I = L\frac{dI}{dt}$$ voltage drop in L

$$V_C = \frac{Q}{C}$$ (charge) voltage drop in C

[Q] Coulombs

$$\frac{dV_c}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{1}{C} I$$

Since $\frac{dQ}{dt} = I$ the current.

$$E(t) = R\frac{dQ}{dt} + \frac{Q}{C} + L\frac{d^2Q}{dt^2}$$

2nd order Linear ode for $Q(t)$

Constant Coefficients

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{E}{L}$$

with solution

$$Q(t) = Q_H + Q_p$$

Use MUC or Variation of parameters to find Q_p

As an IVP

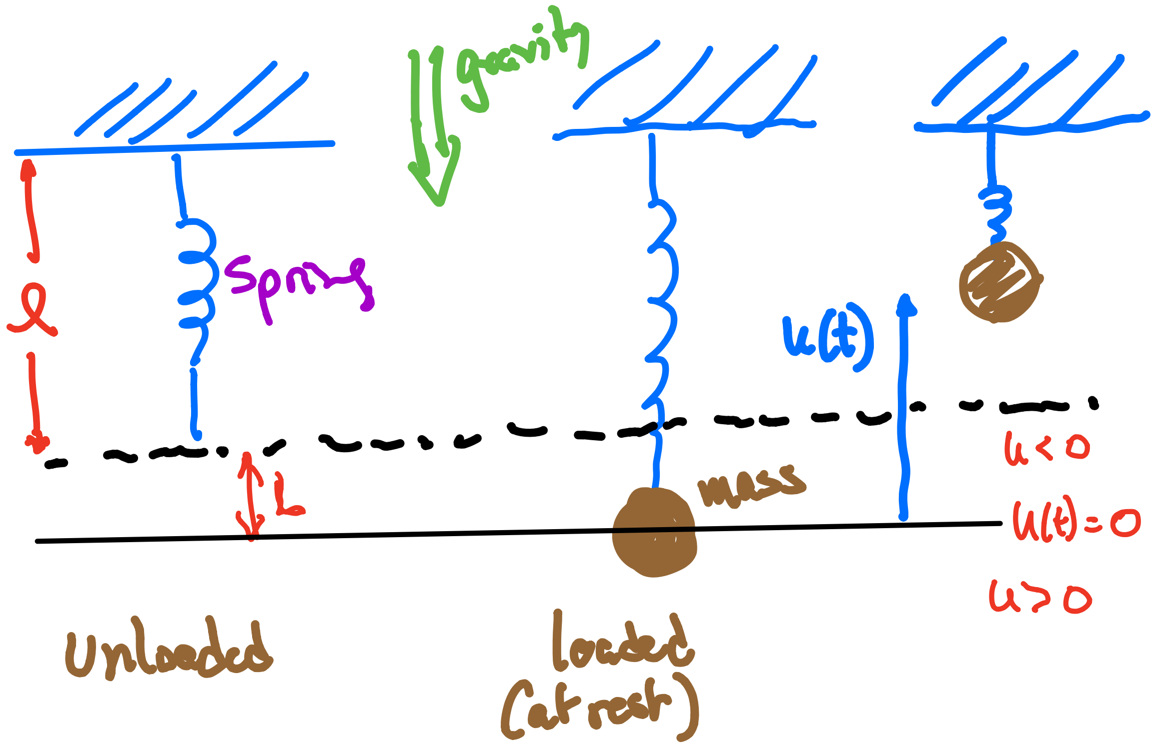
$$\begin{cases} \dfrac{d^2Q}{dt^2} + \dfrac{R}{L}\dfrac{dQ}{dt} + \dfrac{Q}{LC} = \dfrac{E(t)}{L} \quad (\$) \\ \\ Q(t_0) = Q_0 , \quad I(t_0) = \left.\dfrac{dQ}{dt}\right|_{t=t_0} = I_0 \end{cases}$$

If we differentiate ($) since $I = dQ/dt$:

$$\frac{1}{L}\frac{dE}{dt} = \frac{d}{dt}\left[\frac{dI}{dt} + \frac{R}{L}I + \frac{Q}{LC}\right]$$

$$\frac{1}{L}\frac{dE}{dt} = \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I \quad //$$

MECHANICAL VIBRATIONS



Newton's Second Law

Let $u(t)$ be the displacement at time t. The units are $[u(t)]$ = length and $[t]$ = time.

We'll adopt the convention that $u(t)>0$

Force balance:

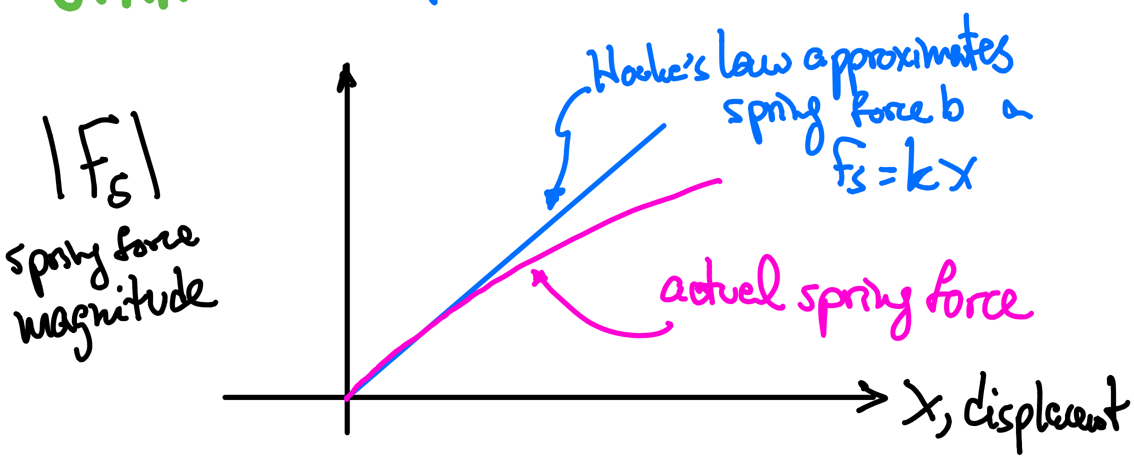
$$(\dagger) \qquad m\,u''(t) = mg - k\,[u(t)+L]$$

$$m\,a \;=$$

mg is the gravitational force

$-k[u(t)+L]$ is the spring force

SPRING FORCE (Hooke's Law)



Hooke's law: assumes that the spring force F_S can be approximated by kx (OK, for small x)

k is the spring constant (slope of linear approximation.

k large "stiff spring"

k small "non-stiff spring"

k has units $[k] =$ force/length

If mass is "at rest", $u = 0$ for all t,

$$mg = kL \qquad (*)$$

if $(*)$ holds then $(\dagger)$ is

$$m u''(t) = -k u(t)$$

or $$u''(t) + \frac{k}{m} u(t) = 0$$

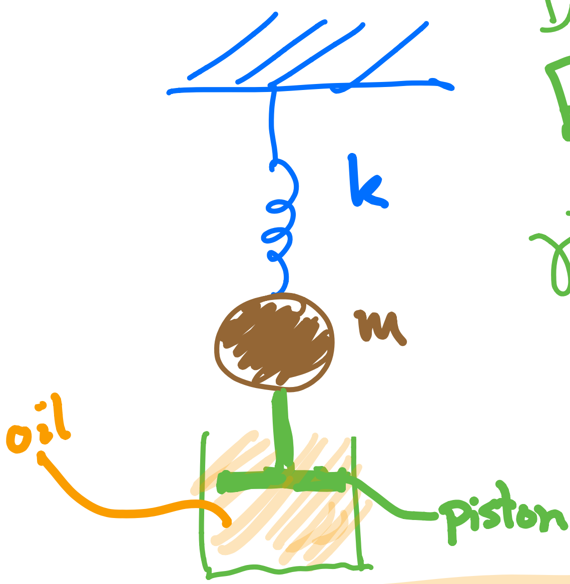
let $\omega = \sqrt{\frac{k}{m}}$ natural frequency

$$\boxed{\therefore \quad u''(t) + \omega^2 u(t) = 0}$$

SHO equation

simple harmonic oscillator equation

Adding Damping Force:



Damping force

$$F_d = -\gamma u'$$

γ is a constant (damping constant)

units $[\gamma]$: $\frac{\text{force}}{\text{velocity}}$

$$u''(t) + \beta u' + \omega^2 u = 0$$

$$\beta = \frac{\gamma}{m} \qquad \omega^2 = \frac{k}{m}$$

Unforced, Damped, simple harmonic osillator

Add initial conditions to form an IVP:

$$u(t_0) = u_0 \quad , \quad u'(t_0) = u_1$$

Add External force $F(t)$:

$$u'' + \beta u' + \omega^2 u = f(t)$$

$$f(t) = \frac{F(t)}{m}$$

$F(t)$ is the external force (given)

Forced Oscillator	RLC
$\beta = \frac{\gamma}{m}$ "damping"	$\beta = \frac{R}{L}$
$\omega^2 = \frac{k}{m}$ "natural freq squared"	$\omega^2 = \frac{1}{LC}$
$f(t) = \frac{F(t)}{m}$ "external forcing"	$f(t) = \frac{E(t)}{L}$
u is displacement	Q charge
$u(t_0)$ initial displacement	$Q(t_0)$ initial charge
$u'(t_0)$ initial velocity	$Q'(t_0)$ initial current

GENERAL SOLUTION:

$$\text{IVP}\begin{cases} y'' + \beta y' + \omega^2 y = f(t), \ t>0 \\ y(0) = Y_0 \ , \ y'(0) = Y_1 \end{cases}$$

$$y(t) = y_H(t) + y_p(t)$$

$$y_H = c_1 y_1(t) + c_2 y_2(t)$$

Rmk: don't forget to find c_1 & c_2 after finding y_p.

HOMOGENEOUS PROBLEM:

Propose solution to $y_H \sim e^{mt}$

$$m^2 + \beta m + \omega^2 = 0$$

characteristic equation

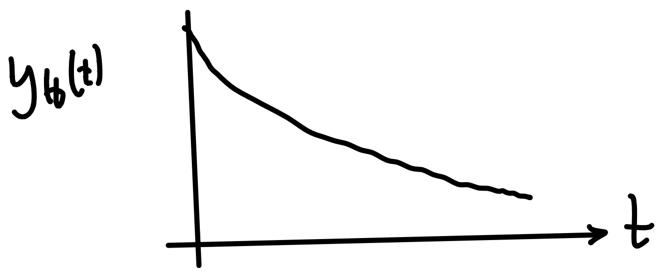
physics dictate that $\beta \geq 0$. $\omega^2 \geq 0$

$$m_{1,2} = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 - 4\omega^2}$$

if $\beta^2 > 4\omega^2$ **OVERDAMPED**

$$y_H = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

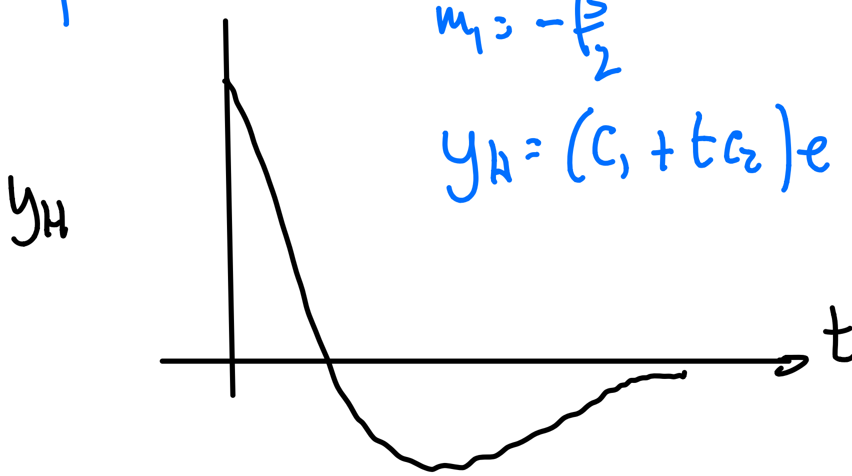
$m_{1,2}$ are real & distinct



if $\beta^2 = 4\omega^2$ **CRITICALLY DAMPED**

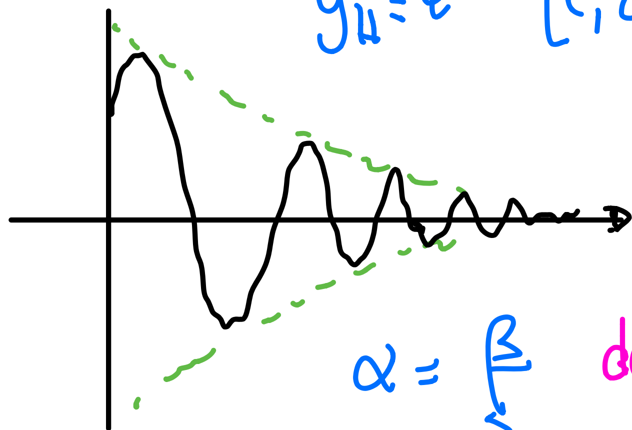
$$m_1 = -\frac{\beta}{2}$$

$$y_H = (c_1 + t c_2) e^{m_1 t}$$



if $\beta^2 < 4\omega^2$ **UNDERDAMPED**

$$y_H = e^{-\alpha t}\left[c_1 \cos\omega_d t + c_2 \sin\omega_d t\right]$$



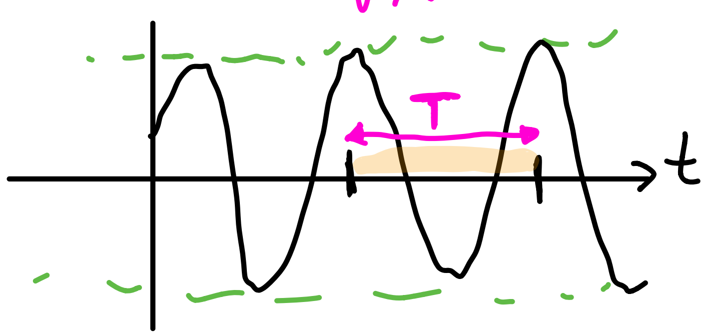
$\alpha = \frac{\beta}{2}$ damping rate

$\omega_d = \sqrt{4\omega^2 - \beta^2}$ damped frequency

if $\beta = 0$ **UNDAMPED**

$$y_H = c_1 \cos\omega t + c_2 \sin\omega t$$

$\omega = \sqrt{\frac{k}{m}}$ the natural frequency



$T = 2\pi/\omega$ period

ex) A mass weighing 2 lbs stretches a spring 6 inches. At $t=0$ the mass is released 8 inches below equilibrium ($u=0$) with an upward velocity of $\frac{4}{3}$ ft/sec. There is no external forcing. Find the displacement as a function of time:

① Find k: when the system is at rest

$$mg = kL \qquad mg = 2\text{lbs}$$

$$L = \frac{1}{2}\text{ ft}$$

$$\therefore \boxed{k = \frac{L}{mg} = \frac{1}{4}\text{ lb/ft}}$$

Rule: in the english system of units weight is a force, hence mg = weight.
The unit of mass is SLUG, the acceleration of gravity is approximately 32 ft/sec^2.

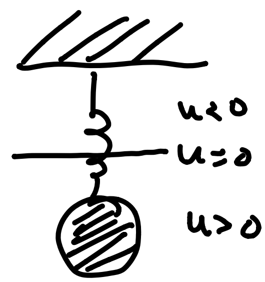
Since no forcing or damping ($\beta = 0$)

$$u'' + \omega^2 u = 0$$

$$\omega^2 = \frac{k}{m} = 64$$

$m = 2\text{lbs} / 32\text{ft/sec}^2 = \frac{1}{16}$ slugs.

$$\begin{cases} u'' + 64u = 0 \\ u(0) = \frac{2}{3} \quad u'(0) = -\frac{4}{3} \end{cases}$$



$u(t) = A\cos 8t + B\sin 8t \qquad \omega = 8$

$u(0) = \frac{2}{3} = A$

$u'(t) = -8A\sin 8t + 8B\cos 8t$

$u'(0) = -\frac{4}{3} = 8B \Rightarrow B = -\frac{1}{6}$

$$\therefore u(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

We can write

$$y = A\cos\omega t + B\sin\omega t$$

$$y = R\cos(\omega t + \delta)$$

$$R = \sqrt{A^2 + B^2}$$

$$\tan\delta = -\frac{B}{A}$$

$$R = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} \approx 0.687$$

$$\tan\delta = \frac{1/6}{2/3} \Rightarrow \delta \approx 0.245$$

$$u(t) = 0.687\cos(8t + 0.245)$$

//

ex) Find the current in an RLC circuit with $L=0.5H$, $R=15\Omega$, $C=2\cdot 10^{-3}F$. The circuit was charged to a steady state $Q(0)=Q_0$. There is no power source after $t=0$.

$$\beta=\frac{R}{L}=\frac{15}{0.5}=30$$

$$\omega^2=\frac{1}{LC}=\frac{1}{2\cdot 0.5\cdot 10^{-3}}=10^{+3}$$

$$\therefore \begin{cases} Q''+30Q'+10^3Q=0 \\ Q(0)=Q_0 \qquad Q'(0)=0 \end{cases}$$

this follows from assuming that the circuit was in a steady state before $t=0$.

$$Q\sim e^{mt} \Rightarrow m^2+30m+10^3=0$$

$$m=-15\pm i5\sqrt{31}$$

$$\therefore Q(t)=e^{-15t}\left[A\cos 5\sqrt{31}\,t+B\sin 5\sqrt{31}\,t\right]$$

$Q(0) = Q_0 = A$

$Q'(0) = 0 \Rightarrow B = 0$

$\therefore Q(t) = \left[e^{-15t} \cos 5\sqrt{31}\, t\right] Q_0$