3.6 Variation of Parameters

This is a general method that generates a particular solution to

$$Ly = g(t)$$

where L is any linear n^{th} order differential operator.

CONSIDER THE LINEAR SECOND ORDER CASE:

$$y'' + p(t)y' + q(t)y = g(t)$$

with solution

$$y = y_H + y_P, \text{ where}$$

$$\begin{cases} y_H'' + p(t)y_H' + q(t)y_H = 0 & (\$) \\ y_P'' + p(t)y_P' + q(t)y_P = g(t) & (£) \end{cases}$$

The method presumes you have already obtained a solution to \$; i.e.

you know

$$y_H = c_1 y_1(t) + c_2 y_2(t)$$

The idea is to propose

$$(*) \quad y_p = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

we know y_1 & y_2.

$u_1(t)$ & $u_2(t)$. Need to find u_1 & u_2:

Replace $(*)$ into $(\not{y})$:

We'll need

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

and $$y_p'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$

So, replacing into $(\not{y})$:

$$\underbrace{\left(u_1''y_1 + 2u_1'y_1' + u_1y_1''\right) + \left(u_2''y_2 + 2u_2'y_2' + u_2y_2''\right)}_{y_p''}$$

$$+\, p(t)\underbrace{\left(u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'\right)}_{y_p'}$$

$$+\, q(t)\underbrace{\left(u_1y_1 + u_2y_2\right)}_{y_p} = g(t)$$

$$u_1(y_1'' + py_1' + qy_1) = 0$$

$$u_2(y_2'' + py_2' + qy_2) = 0$$

$$\therefore\ (u_1'y_1 + u_2'y_2)' + u_1'y_1' + u_2'y_2'$$

$$+\, p(t)(u_1'y_1 + u_2'y_2) = g(t) \qquad (*)$$

$$(u_1''y_1 + u_1'y_1' + u_2''y_2 + u_2'y_2')$$

Set (A) $u_1' y_1 + u_2' y_2 = 0$ in $(\maltese)$

then (B) $u_1' y_1' + u_2' y_2' = g(t)$.

Rewrite (A) & (B) as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

Solve for u_1' and u_2' using Cramer's Rule:

$$(\maltese) \begin{cases} u_1' = \dfrac{1}{W} \det \begin{bmatrix} 0 & y_2 \\ g(t) & y_2' \end{bmatrix} = -\dfrac{1}{W} y_2 g(t), \\ u_2' = \dfrac{1}{W} \det \begin{bmatrix} y_1 & 0 \\ y_1' & g(t) \end{bmatrix} = \dfrac{1}{W} y_1 g(t), \end{cases}$$

where W = Wronskian of y_1 & y_2

$$W = y_1 y_2' - y_1' y_2$$

Rmk: $W \neq 0$ for $t \in I$ $\therefore$

u_1' & u_2' can be found.

Last step: integrate u_1' & u_2' to find u_1 and u_2. Then the desired result is

$$y_p = u_1 y_1 + u_2 y_2$$

ex) $\quad y'' - 2y' - 3y = -3te^{-t}$

Try using MUC, to get

$$y = c_1 e^{-t} + c_2 e^{3t} + \underbrace{\left(\frac{3}{8}t^2 + \frac{3}{16}t + \frac{3}{64}\right)e^{-t}}_{y_p}$$

So $\quad y_1 = e^{-t} \quad , \; y_2 = e^{3t}$

Let's see if we obtain the same result via variation of parameters:

let $\quad y_p = u_1(t)\, y_1 + u_2(t)\, y_2$

Use $(\ast)$ to find u_1' & u_2'

$$W = \det\begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = 3e^{2t} + e^{2t} = 4e^{2t}$$

$$u_1' = -\frac{1}{W}\underset{\substack{\uparrow \\ y_2}}{e^{3t}}\underset{\substack{\uparrow \\ g}}{\left(-3te^{-t}\right)} = \frac{3t}{W}e^{2t} = \frac{3}{4}t$$

$$u_2' = \frac{1}{W}\underset{\substack{\uparrow \\ y_1}}{e^{-t}}\underset{\substack{\uparrow \\ g}}{\left(-3te^{-t}\right)} = -\frac{3te^{-2t}}{W} = -\frac{3t}{4}e^{-4t}$$

$$u_1 = \int \frac{3}{4}t\,dt = \frac{3}{8}t^2$$

$$u_2 = -\frac{3}{4}\underset{\text{IBP}}{\int te^{-4t}dt} = \frac{3}{8}\left(\frac{t}{2} + \frac{1}{8}\right)e^{-4t}$$

$$y_p = \frac{3}{8}t^2e^{-t} + \left(\frac{3t}{16}e^{-4t} + \frac{3}{64}e^{-4t}\right)e^{3t}$$

$$y_p = \frac{3}{8}t^2 e^{-t} + \frac{3t}{16}e^{-t} + \frac{3}{64}e^{-t}$$

So it agrees with the MUC result.

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