SUMMARY OF SOLUTIONS TO HOMOGENEOUS 2nd ORDER ODES

$$Ly = g(x)$$

has a solution $y = \underbrace{c_1 y_1 + c_2 y_2}_{y_H} + y_p(x)$

where $Ly_1 = 0 \quad Ly_2 = 0 \quad Ly_p = g(x)$

CONSTANT COEFFICIENT 2nd Order ODE:

$$y'' + \alpha y' + \beta y = 0$$

$$y_H = e^{mx}$$

$$\therefore \quad m^2 + \alpha m + \beta = 0$$

$$m_{1,2} = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$$

$$y_h = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

if $\alpha^2 - 4\beta > 0$ $\quad m_{1,2}$ are distinct real roots

$$y_h = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

if $\alpha^2 = 4\beta$ $\quad m_{1,2} = -\frac{\alpha}{2} = m$

$$y = (C_1 + C_2 x) e^{mx}$$

if $\alpha^2 - 4\beta < 0$

$$y_h = e^{-\frac{\alpha}{2}x}\left(A\cos\omega x + B\sin\omega x\right)$$

where $\quad \omega = \sqrt{4\beta - \alpha^2}$

2nd order Euler Equations

$$x^2 y'' + \alpha x y' + \beta y = 0$$

$$y = x^m$$

$$m(m-1) + \alpha m + \beta = 0$$

$$m^2 + (\alpha-1)m + \beta = 0$$

$$m_{1,2} = -\frac{\alpha-1}{2} \pm \frac{1}{2}\sqrt{(\alpha-1)^2-4\beta}$$

if $(\alpha-1)^2-4\beta > 0$ then

$$y_h = C_1 x^{m_1} + C_2 x^{m_2}$$

$m_{1,2}$ are distinct real roots

if $(\alpha-1)^2 = 4\beta$ then $m_{1,2} = m = \frac{1-\alpha}{2}$

$$y_h = (C_1 + C_2 \ln x)e^{mx}$$

if $4\beta > (\alpha-1)^2$ then $m_{1,2}$ are complex conjugate roots

$$y_h = x^{-\frac{(\alpha-1)}{2}}\left[A\cos(q\ln x) + B\sin(q\ln x)\right]$$

where $q = \frac{1}{2}\sqrt{4\beta-(\alpha-1)^2}$

//

3.5 Non-Homogeneous Equations

The solution to

$$Ly = g(x)$$

where L is a linear differential operator

is

$$y = y_h + y_p$$

where $Ly_h = 0$

$$Ly_p = g(x)$$

How do we find y_p?

We will learn two methods. They both apply to linear differential equations.

① Method of Undetermined Coefficients (MUC)

Applies only to C.C. ODE's!!

② Variation of Parameters

Applies to general linear ODE's, including C.C. ODE's.

METHOD OF UNDETERMINED COEFFICIENTS

The basic idea is to GUESS the form of solution

Given $Ly = g(x)$

$g(x)$	form of guess $y_p(x)$
$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$	$x^s(A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x + A_0)$
$P_n(x)\, e^{\alpha x}$ (α is a constant)	$x^s(A_n x^n + \dots A_1 x + A_0)e^{\alpha x}$
$P_n(x) e^{\alpha x} \begin{cases} \sin\beta x \\ \cos\beta x \end{cases}$	$E(x)$

where

$$E(x) = x^s(A_n x^n + A_{n-1}x^{n-1} \dots A_0)\, e^{\alpha x}\sin\beta x$$

$$+ x^s(B_n x^n + B_{n-1}x^{n-1} \dots B_0)\, e^{\alpha x}\cos\beta x$$

Rmk: in this last case ALWAYS use a combination of sin and cos, regardless of whether $g(x)$ contains only sin or only cos (or both).

Rmk: the term x^s will be explained after we consider what can go wrong with the method of guessing (MUC).

S is an integer 0, 1, 2... that will ensure that NO term in $y_p(x)$ is a solution of $Ly_H = 0$

Rmk: if $g(x)$ is a linear combination, then make $y_p(x)$ be a guess consisting of a linear combination of the assumed guesses.

ex)

$$Ly = g(t) \qquad y = y(t)$$

$$y'' + 6y' + 9y = 3e^{2t}$$

General Solution $y(t) = y_H(t) + y_p(t)$, where

$$y_H'' + 6y_H' + 9y_H = 0$$

$$y_H \sim e^{mt}$$

$$m^2 + 6m + 9 = 0 \Rightarrow m = -3$$

$$y_H = c_1 e^{-3t} + c_2 t e^{-3t}$$

reduction of order process.

Solve $Ly_p = g(t)$ (*) use MUC

here, $g(t) = 3e^{2t}$ so assume

$$y_p = Ae^{2t}$$

A is unknown but will be found by forcing y_p to satisfy (*).

Need these: $y_p' = 2Ae^{2t}$ $y_p'' = 4Ae^{2t}$, inserting in (*):

$$\underbrace{4Ae^{2t}}_{y_p''} + 6\underbrace{(2Ae^{2t})}_{y_p'} + 9\underbrace{(Ae^{2t})}_{y_p} = 3e^{2t}$$

$$A[4 + 12 + 9]e^{2t} = 3e^{2t}$$

solve for $A = \frac{3}{25}$

$$\therefore \quad y_p = \frac{3}{25}e^{2t}$$

$$y = c_1 e^{-3t} + c_2 t e^{-3t} + \frac{3}{25}e^{2t} \quad //$$

ex) $y'' - 6y' + 9y = -12e^{3t}$

$$y_H = Ae^{3t} + Bte^{3t}$$

$$y_p = Ce^{3t} + (D_0 + D_1 t)e^{3t}$$

This guess is not good because

because $Ly_p = 0$.

We use t^s (guess) to make

y_p form Linearly independent of y_H:

Use $y_p = t^{s=1}(C_0 + C_1 t)e^{3t} = (C_0 t + C_1 t^2)e^{3t}$

substitute into $y_p'' - 6y_p' + 9y_p = -12e^{3t}$:

You get $c_0 = 0$ and $2c_1 + 12 = 0$

$\therefore c_1 = -6$

$y_p = -6t^2 e^{3t}$ //

ex) $y'' - 5y' + 4y = 8te^{6t}$

$m^2 - 5m + 4 = 0 \Rightarrow m_{1,2} = 1, 4$

$y_H = c_1 e^t + c_2 e^{4t}$

$y_p = (At + B)e^{6t}$

$$y_p' = 6(At+B)e^{6t} + Ae^{6t}$$

$$y_p'' = 36(At+B)e^{6t} + 6Ae^{6t} + 6Ae^{6t}$$

Substitute into $Ly_p = 8te^{6t}$

$$36(At+B) + 12A - 5\big(6(At+B) + A\big) + 4(At+B) = 8t$$

Solve for A & B:

Collect terms in powers of t:

$$36A - 30A + 4A = 8 \qquad : t^1$$

$$36B + 12A - 5B - 5A + 4B = 0 \qquad : t^0$$

$$A = \frac{4}{5} \qquad B = -\frac{28}{50}$$

$$y_p = \left(\frac{4}{5}t - \frac{28}{50}\right)e^{6t}$$

ex) $y'' + y = 4t + 10\sin t$

$m^2 + 1 = 0 \qquad m = \pm i$

$y_H = A\sin t + B\cos t$

Solve $\quad L y_p = 4t + 10\sin t = g(t) \quad (\ast)$

$y_p = C_1 t + C_2$

$t^s = t^1$

$+ (D\cos t + E\sin t)\, t$

substitute into $(\ast)$

$y_p' = C_1 + (D\cos t + E\sin t)$
$\quad + t(-D\sin t + E\cos t)$

$y_p'' = -D\sin t + E\cos t + (-D\sin t + E\cos t)$
$\quad [-D\cos t - E\sin t]\, t$

$$y_p'' = -2D\sin t + 2E\cos t - t[D\cos t + E\sin t]$$

$$\overbrace{-2D\sin t + 2E\cos t - \cancel{t[D\cos t + E\sin t]}}^{y''}$$

$$+ \underbrace{c_1 t + c_2 + \cancel{t[D\cos t + E\sin t]}}_{y}$$

$$= 4t + 10\sin t$$

$$-2D\sin t + 2E\cos t + c_1 t + c_2 = 4t + 10\sin t$$

$$\therefore \quad E = 0$$

$$c_2 = 0$$

$$c_1 = 4$$

$$-2D = 10 \Rightarrow D = -5$$

$$\therefore y_p = 4t - 5\sin t$$

ex) $y'' + y = 2x\sin x$ $\qquad y = y(x)$

$$y = C_1\cos x + C_2 \sin x + y_p$$

$$y_p = (a_0 + a_1 x + a_2 x^2)\cos x + (b_0 + b_1 x + b_2 x^2)\sin x$$

substituting into $y'' + y = 2x\sin x$:

$$4b_2 x\cos x - 4a_2 x\sin x + 2a_2\cos x + 2b_1\cos x$$
$$- 2a_1 \sin x + 2b_2 \sin x - 2x\sin x = 0$$

$\Rightarrow \quad a_0 = 0 \qquad b_0 = 0$

$-4a_2 - 2 = 0 \quad \Rightarrow a_2 = \frac{1}{2}$

$4b_2 = 0 \quad \Rightarrow b_2 = 0$

$2a_2 + 2b_1 = 0 \quad \Rightarrow b_1 = -a_2 = -\frac{1}{2}$

$-2a_1 + 2b_2 = 0 \quad \Rightarrow a_1 = 0$

$\therefore y_p = \frac{1}{2}x^2\cos x - \frac{1}{2}x\sin x$ //

1x) $y'' + 4y' + 4y = (3+x)e^{-2x}$ (☆)

$y(0) = 2 \quad y'(0) = 5$

$y = y_h + y_p$

Apply I.C. to $y = y_h + y_p$ (not just y_h.

$y_H'' + 4y_H' + 4y_H = 0$

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$\Rightarrow y_H = (c_1 + c_2 x)e^{-2x}$

So Ⓐ $y_p = x^2 (b_0 + b_1 x) e^{-2x}$

Ⓑ $y_p' = 2x(b_0 + b_1 x)e^{-2x}$
$+ x^2 (b_1) e^{-2x} - 2x^2 (b_0 + b_1 x) e^{-2x}$

Ⓒ $y_p'' = 2(b_0 + b_1 x)e^{-2x} + 2b_1 x e^{-2x} - 4x(b_0 + b_1 x)e^{-2x}$
$+ 2x b_1 e^{-2x} - 2b_1 x^2 e^{-2x} - 4x(b_0 + b_1 x)$
$- 2x^2 b_1 e^{-2x} + 4x^2 (b_0 + b_1 x) e^{-2x}$

Replace Ⓐ, Ⓑ, Ⓒ into ☆

Get: $[2(b_1 x + b_0) + 4xb_1]e^{-2x} = (3+x)e^{-2x}$

$\therefore\ 2b_0 = 3 \qquad \therefore\ b_0 = \frac{2}{3}$

$2b_1 + 4b_1 = 1 \qquad b_1 = \frac{1}{6}$

$$y = (c_1 + c_2 x)e^{-2x} + \left(\frac{2}{3} + \frac{x}{6}\right)e^{-2x}$$

Apply I.C.

$y(0) = 2 = c_1 + \frac{2}{3} \Rightarrow c_1 = \frac{4}{3}$

$y' = c_2 e^{-2x} - 2(c_1 + c_2 x)e^{-2x} + \frac{1}{6}e^{-2x} - 2\left(\frac{2}{3} + \frac{x}{6}\right)e^{-2x}$

$y'(0) = 5 = c_2 - 2c_1 + \frac{1}{6} - \frac{4}{3}$

$5 = c_2 - \frac{8}{3} + \frac{1}{6} - \frac{4}{3} = c_2 - 4 + \frac{1}{6}$

$5 = c_2 - \frac{23}{6} \qquad -\frac{24}{6} + \frac{1}{6} = -\frac{23}{6}$

$\therefore c_2 = 5 + \frac{23}{6} = \frac{53}{6}$

$\therefore y = \left(\frac{4}{3} + \frac{53}{6}x\right)e^{-2x} + \left(\frac{2}{3} + \frac{x}{6}\right)e^{-2x}$

$= \frac{5}{3}e^{-2x} + \frac{27}{2}xe^{-2x}$ //