3.4 Repeated Roots

Focus on homogeneous C.C. first:

$$y'' + \alpha y' + \beta y = 0 \quad (\$), \quad \alpha, \beta \text{ constants}$$

the solution is $y = c_1 y_1(x) + c_2 y_2(x)$
where y_1, y_2 are the linearly-independent fundamental solutions.

But suppose $m^2 + \alpha m + \beta = 0$
with roots

$$m_{1,2} = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$$

generates repeated roots, i.e. when

$\alpha^2 = 4\beta$. Then $m_{1,2} = -\frac{\alpha}{2}$.

Hence we can get $y_1(x) = e^{-\frac{\alpha}{2}x}$.

How do we find $y_2(x)$?

We want to find $y_2(x)$ which is a solution to $(\$)$ AND is linearly-independent.

We'll use the following procedure:

The Reduction of Order Process.

Before introducing the process, let's be sure we know what is meant by linearly-independent:

Def: The functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be linear independent on some interval I iff the constants $c_1, c_2, \ldots, c_n$ must ALL BE ZERO to satisfy

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \text{, on } I$$

Consider $f_1(x)$ and $f_2(x)$. Test to see if f_1 & f_2 are linearly independent. We'd look to see if the only constants that satisfy $c_1 f_1(x) + c_2 f_2(x) = 0$ are $c_1 = c_2$ over some I.

What happens when f_1 and f_2 are linearly dependent?

in that case $f_1(x)$ can be written as a nonzero constant times $f_2(x)$ (or viceversa):

hence: $f_1(x) = -\frac{c_2}{c_1} f_2(x) \equiv \alpha f_2(x)$, $\alpha \neq 0$.

whence we say $f_1(x)$ and $f_2(x)$ are NOT linearly independent!

REDUCTION OF ORDER: general procedure that generates a solution, if you know $n-1$ others where n is the order of the linear ODE.

For 2nd order $n=2$. So we assume we know $y_1(x)$ and want to find $y_2(x)$.

Return to ($), assume the repeated root case: Found $y_1(x)$. Build y_2 to be automatically linearly independent:

Take $y_2(x) = u(x)\, y_1(x)$, substitute into ($) and find $u(x)$.

For $y'' + p(x)\,y' + q(x)\,y = 0$ (A)

Spse we know $y_1(x)$.

let $y_2(x) = u(x)\,y_1(x)$ Substitute into (A):

$$\begin{cases} y_2(x) = u\,y_1 \\ y_2'(x) = u'y_1 + u\,y_1' \\ y_2'' = u''y_1 + u'y_1' + u'y_1' + u\,y_1'' \end{cases}$$

To get:

$$\left(u''y_1 + 2u_1'y_1' + u\,y_1''\right) + p(x)\left(u'y_1 + u\,y_1'\right) + q(x)\,u\,y_1 = 0.$$

We note that the ✓ add upto zero:

✓ $\left(y_1'' + p(x)y_1' + q(x)y_1\right)u = 0$. What remains:

$$u_1''y_1 + 2u_1'y_1' + pu'y_1 = 0 \quad (\text{Ŧ})$$

divide (Ŧ) by y_1:

$$u_1'' + 2u_1'\frac{y_1'}{y_1} + pu' = 0$$

or $$u_1'' + \left[p + \frac{2y_1'}{y_1}\right]u_1' = 0$$

Let $w = u'$

$$\therefore \quad w' + \left[p + \frac{2y_1'}{y_1}\right]w = 0 \quad (\#)$$

a separable ODE (we know p, y_1 & y_1')

1. Solve for w in (#)
2. Integrate w to find u
3. form $y_2 = uy_1$.

Note: $$w' = -\left[p + \frac{2y_1'}{y_1}\right]w \qquad (\#), \text{ or}$$

$$\frac{dw}{w} = -\left[p + \frac{2y_1'}{y_1}\right]dx$$

integrate and exponentiate, to get

$$(\not{\#}) \quad w = e^{-\int\left[p + \frac{2y_1'}{y_1}\right]dx} = \frac{1}{y_1^2}e^{-\int p\,dx}$$

to see this:

$$w = e^{-\int p dx}e^{-2\int \frac{y_1'}{y_1}dx} = e^{-\int p dx}e^{-2\ln y_1} = e^{-\int p dx}e^{\ln y_1^{-2}}$$

$$= \frac{1}{y_1^2}e^{-\int p dx} \qquad //$$

ex) $y'' + 4y' + 4y = 0$, e.c. example

$y = e^{mx}$ $\qquad m^2 + 4m + 4 = 0$

or $(m+2)^2 = 0$ (repeated root!)

$\therefore\ y_1 = e^{-2x}$ use Reduction of order:

let $y_2 = u(x)\, y_1$

Using $(\mathcal{Y})$ here $p = 4$

$$W = \frac{1}{y_1^2} e^{-\int 4\,dx} = e^{4x} e^{-4x} = 1$$

$$u = \int w\,dx = \int dx = x$$

$$\therefore y_2 = x y_1 = x e^{-2x}$$

$$\therefore y = c_1 x e^{-2x} + c_2 e^{-2x}$$

ex) An Euler Equation example:

(*) $$x^2 y'' - (2a-1) x y' + a^2 y = 0$$

here a is number

$$y \sim x^m \Rightarrow (m-a)^2 = 0$$

repeated root. Use reduction of order:

$y_1 = x^a$ find $y_2(x) = u(x) y_1$

First, write (*) in the form (☆):

$$y'' - \frac{(2a-1)}{x^2} x y' + \frac{a^2}{x^2} y = 0$$

$$\therefore p(x) = -\frac{(2a-1)}{x}$$ (see ☆)

$$W = \frac{1}{y_1^2} e^{-\int p\,dx} = \frac{1}{y_1^2} e^{(2a-1)\int \frac{dx}{x}}$$

$$= \frac{1}{y_1^2} e^{(2a-1)\ln x}$$

$$= \frac{1}{y_1^2} e^{\ln x^{(2a-1)}} = \frac{1}{y_1^2} x^{2a-1}$$

$W = x^{-2a} x^{2a-1} = \frac{1}{x} = u'$, integrating:

$u = \ln|x| \therefore y_2 = \ln|x|\, x^a$

General solution to (*) is thus

$$y = x^a (c_1 + c_2 \ln|x|)$$