3.3 Complex Roots Case for C.C. & E.E. 2nd Order ODEs

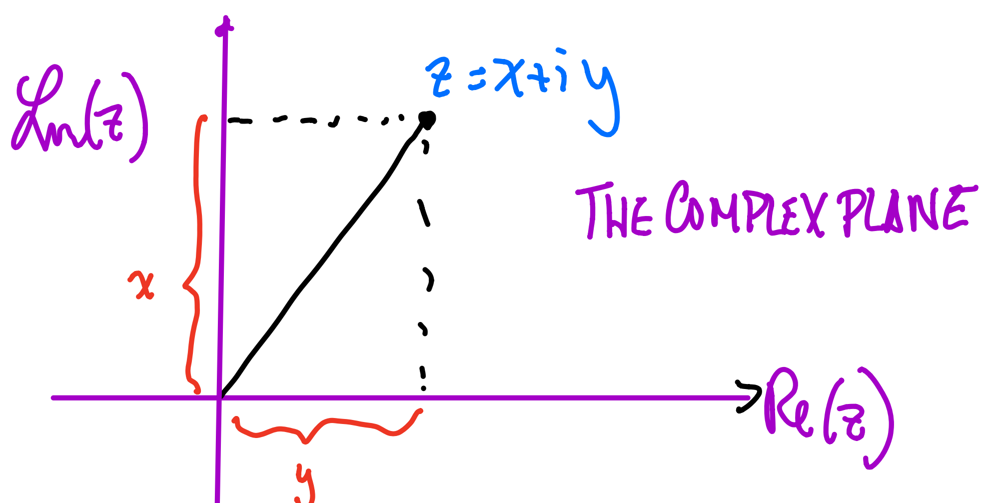
Short Review of Complex Numbers

Let $z = x + iy$ $\quad w = p + iq$

where $i = \sqrt{-1}$ "Cartesian Representation"

x, y, p, q are real #'s. //

The complex conjugate $\bar{z} = x - iy$



Modulus $|z| = \sqrt{x^2+y^2}$ also found by:

$$|z| = \sqrt{z\bar{z}}$$

$$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + iyx - ixy + y^2$$

$$= x^2 + y^2$$

PROPERTIES OF $\sqrt{-1}$:

$i = \sqrt{-1}$ $\qquad$ $\frac{1}{i} = -i$

$i^2 = -1$ $\qquad$ & $i(-i) = -(i)^2 = -(-1) = 1$

$i^3 = i^2 i = -1 i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

Adding/Subtracting #'s:

$$z \pm w = (x+iy) \pm (p+iq)$$

$$= (x \pm p) + i(y \pm q)$$

Multiplying Complex #s:

$$z \cdot w = (x+iy)(p+iq)$$

$$= xp + iyp + iqx + (iy)(iq)$$

$$= xp + i(yp + qx) - 1yq$$

$$= (xp - yq) + i(yp + qx)$$

Dividing Complex #:

$$\frac{z}{w} = \frac{z}{w}\,\frac{\bar{w}}{\bar{w}}$$

multiply top & bottom by complex conjugate of denominator

$$w\bar{w} = (p+iq)(p-iq) = p^2 + iq(-iq)$$

$$\therefore \quad w\bar{w} = p^2 + q^2$$

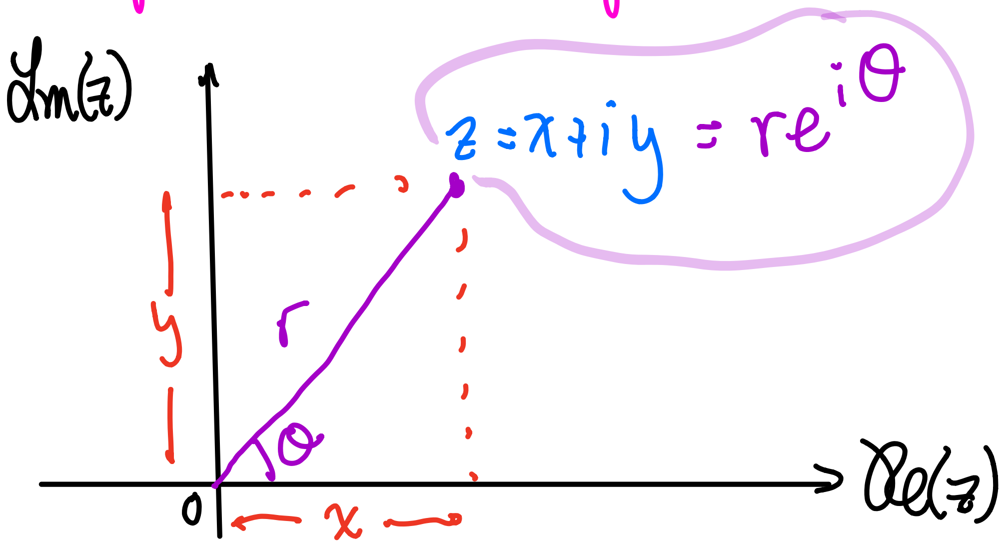
$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{1}{p^2+q^2}\, z\bar{w}$$

$$= \frac{1}{p^2+q^2}(x+iy)(p-iq)$$

$$= \frac{1}{p^2+q^2}\left[xp + iyp - ixq + yq\right]$$

$$= \frac{1}{p^2+q^2}\left[(xp+yq)+i(yp-xq)\right]$$

Polar Representation of Complex Numbers



MAPPING BETWEEN CARTESIAN & POLAR:

$$r = \sqrt{x^2+y^2} \qquad x = r\cos\theta$$

$$\tan\theta = \frac{y}{x} \qquad y = r\sin\theta$$

Rmk: Add/subtracting cartesian complex is easy.
Multiplying/Dividing polar complex is easy.

"phase"

ex) $z = r_1 e^{i\theta_1}$ $\quad w = r_2 e^{i\theta_2}$

$$zw = r_1 r_2 e^{i(\theta_1+\theta_2)}$$ easy

$$z/w = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}$$ easy

How do we add/subtract? Use Euler Identities...

EULER IDENTITIES

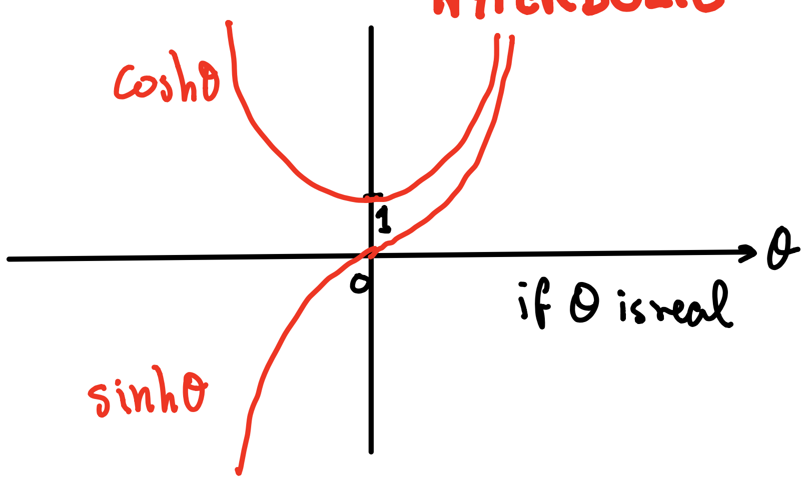
$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

TRIGONOMETRIC

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

HYPERBOLIC



if θ is real

Some properties of hyperbolic functions:

$\cosh\theta$ is even $\qquad$ $\sinh\theta$ is odd

$\cosh(0) = 1$ $\qquad$ $\sinh(0) = 0$

$\lim_{\theta\to\pm\infty} \cosh\theta = \infty$ $\qquad$ $\lim_{\theta\to\pm\infty} \sinh\theta = \pm\infty$

for θ large $\cosh\theta \sim e^{\theta}$, $\sinh\theta \sim e^{\theta}$

for θ negative and large $\cosh\theta \sim e^{\theta}$, $\sinh\theta \sim -e^{-\theta}$

$$\frac{d}{d\theta}\cosh\theta = \sinh\theta \qquad \frac{d}{d\theta}\sinh\theta = \cosh\theta$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$$

ex) Convert $z = 3e^{i2t}$ to cartesian form:

$z = x+iy$ $\qquad$ $x = 3\cos 2t$ $\quad$ $y = 3\sin 2t$

$z = 3(\cos 2t + i\sin 2t)$

ex) Convert $z = 3+2i$ to polar:

$$\left.\begin{array}{l} r = \sqrt{3^2+2^2} = \sqrt{13} \\ \theta = \arctan\left(\frac{2}{3}\right) \end{array}\right\} \quad z = \sqrt{13}\, e^{i\arctan\left(\frac{2}{3}\right)}$$

Back to 2nd order linear C.C.

$$y'' + \alpha y' + \beta y = 0$$

α, β are constants.

The fundamental solution is

$$y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

where $m_{1,2} = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$.

If $\alpha^2 < 4\beta$ then $m_{1,2}$ are complex conjugate pair

$$(\#) \quad y = \left(c_1 e^{+\frac{i}{2}\sqrt{4\beta - \alpha^2}\, t} + c_2 e^{-\frac{i}{2}\sqrt{4\beta - \alpha^2}\, t}\right) e^{-\frac{\alpha}{2} t}$$

c_1 & c_2 are complex coefficients.

This solution is of the form

$$y = \left(c_1 e^{i\theta} + c_2 e^{-i\theta}\right) e^{-\frac{\alpha}{2} t}$$

We could use Euler's identities on $c_1 e^{i\theta}$ & $c_2 e^{-i\theta}$:

If we do, we obtain:

$$y = (A\cos qt + B\sin qt)e^{pt}$$

where $\begin{cases} q = \frac{1}{2}\sqrt{4\beta - \alpha^2} \\ p = -\frac{1}{2}\alpha \end{cases}$ $\quad m_{1,2} = p \mp iq$

& A, B are real constants.

ex) $y'' + y' + y = 0$

$y \sim e^{mt}$ then the characteristic

equation is $m^2 + m + 1 = 0$.

with roots $\quad m_{1,2} = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$

$$\therefore \quad y = c_1 e^{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)t}$$

Can be written as

$$y = \left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)e^{-\frac{1}{2}t} \quad //$$

An alternative representation to 2nd order Euler Equations with complex roots:

Recall $\quad x^2y'' + \alpha x y' + \beta y = 0 \quad$ E.E.

$$\star \quad \begin{cases} y \sim x^m \\ y' = m x^{m-1} \\ y'' = m(m-1)x^{m-2} \end{cases}$$

Sub ☆ into E.E.

$$x^2 m(m-1)x^{m-2} + \alpha x m x^{m-1} + \beta x^m = 0$$

$$m(m-1) + \alpha m + \beta = 0$$

$$m^2 - m + \alpha m + \beta = 0$$

$$m^2 + m(\alpha-1) + \beta = 0$$

with roots

$$m_{1,2} = -\frac{\alpha-1}{2} \pm \frac{1}{2}\sqrt{(\alpha-1)^2 - 4\beta}$$

$$m_{1,2} = \frac{1-\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 2\alpha + 1 - 4\beta}$$

Suppose that $4\beta > (\alpha-1)^2$ so we get complex roots:

$$\therefore\ m_{1,2} = p \pm iq$$

where $p = \frac{1-\alpha}{2}$ $\quad q = \frac{1}{2}\sqrt{4\beta - (\alpha-1)^2}$

$$y = c_1 x^{p+iq} + c_2 x^{p-iq} = x^p\left(c_1 x^{iq} + c_2 x^{-iq}\right)$$

here c_1, c_2 would be complex

FIND A REAL REPRESENTATION

Consider $x^{A \pm iB} = x^A x^{\pm iB} = x^A e^{\ln(x^{\pm iB})}$

$$= x^A e^{\pm iB \ln x}$$

$$x^{A \pm iB} = x^A \left[\cos B\ln x \pm i \sin B \ln x\right]$$

$\therefore$ Euler Equation solution is then

$$\boxed{y = \left[A\cos(q\ln x) + B\sin(q\ln x)\right] x^p}$$

ex) $x^2y'' + 3xy' + 3y = 0$

$y(1) = 1 \quad y'(1) = -5$

$y = x^m$

$x^2 m(m-1)x^{m-2} + 3xmx^{m-1} + 3x^m = 0$

$m(m-1) + 3m + 3 = 0$

$m^2 + 2m + 3 = 0$

$m_{1,2} = -1 \pm \frac{1}{2}\sqrt{4 - 4\cdot 3}$

$= -1 \pm \sqrt{1-3} = -1 \pm i\sqrt{2}$

$\therefore \; y = \left[A\cos(\sqrt{2}\ln x) + B\sin(\sqrt{2}\ln x)\right]x^{-1}$

Apply I.C. (to do so we need y':

$y' = -x^{-2}\left[A\cos(\sqrt{2}\ln x) + B\sin(\sqrt{2}\ln x)\right]$

$$+x^{-1}\left[-\frac{A\sqrt{2}}{x}\sin(\sqrt{2}\ln x)+\frac{B\sqrt{2}}{x}\cos(\sqrt{2}\ln x)\right]$$

$$y(1)=1=\left[A\cos(\sqrt{2}\ln 1)+B\underbrace{\sin\sqrt{2}\ln 1}_{=0}\right]\frac{1}{1}$$

$$1=A$$

$$y'(1)=5=-\left[\cos(\underbrace{\sqrt{2}\ln 1}_{=1})\right]+B\sqrt{2}\cos(\underbrace{\sqrt{2}\ln 1}_{=1})$$

$$\therefore\ 5=-1+B\sqrt{2}$$

$$B=6/\sqrt{2}$$

$$\therefore y=\frac{1}{x}\left[\cos\sqrt{2}\ln x+\frac{6}{\sqrt{2}}\sin\sqrt{2}\ln x\right]$$