SECTION 3.2 EXISTENCE & UNIQUENESS OF SOLUTIONS TO 2nd ORDER LINEAR IVP

Consider:

IVP
$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) & \text{ODE} \\ y(t_0) = Y_0 \quad , \; y'(t_0) = Y_1 & \text{I.C.} \end{cases}$$

Y_1 & Y_2 are given numbers.

A unique solution to IVP exists in $t \in (\alpha, \beta)$ where $t_0 \in (\alpha, \beta)$ if p, q, g are continuous for $t \in (\alpha, \beta)$ //

ex) $2\cos^2 t \, y'' + y = 5\tan t$,

divide by $2\cos^2 t$

$(\star)$ $$y'' + \frac{1}{2}\sec^2(t)\, y = \frac{5}{2}\sec^2 t \tan t$$

here $p = 0$ $\quad q = \frac{1}{2}\sec^2 t$

$g = \frac{5}{2}\sec^2 t \tan t$ (see ODE, above).

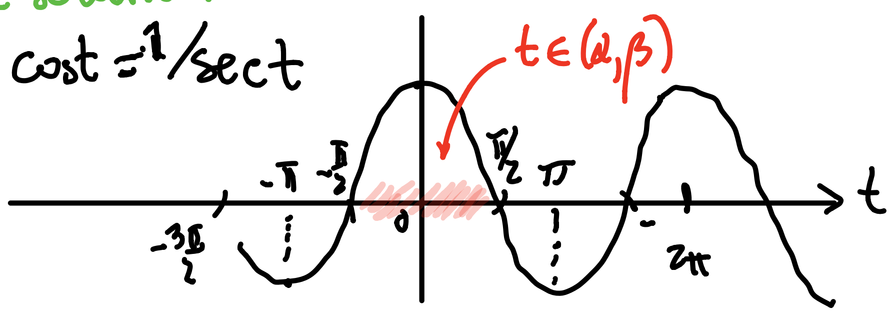
Suppose we are also given the I.C.

$Y_0 = 0 \qquad Y_1 = 1$

$y(0) = 0 \quad , \; y'(0) = 1 .$

Find the range of t values which guarantee a unique solution.

Note: $\cos t = 1/\sec t$



goes to zero at $\pm \frac{\pi}{2}(n+1) \qquad n = 0, 1 \ldots$

$\therefore \quad \sec^2 t$ is undefined there.

Also $\tan t$ is undefined at $\pm \frac{\pi}{2}$

$\therefore$ choose $t \in (\alpha, \beta) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ since it contains $t_0 = 0$

It turns out that we can solve IVP analytically: A solution $y = \tan(t)$ satisfies the ODE.

Check: $y' = \sec^2 t$ $\quad y'' = 2\sec^2 t \tan t$

substitute y, y', y'' into $(\star)$

$$\underbrace{2\sec^2 t \tan t}_{y''} + \underbrace{\tfrac{1}{2}\sec^2 t \tan t}_{\frac{1}{2}\sec^2 t \, y} = \frac{5}{2}\sec^2 t \tan t \quad \checkmark$$

it also satisfies the i.c.:

$$y(0) = \underbrace{\tan(0)}_{y(0)} = 0 \quad \checkmark$$

$$y'(0) = \underbrace{\sec^2 t \Big|_{t=0}}_{y'(0)} = 1 \quad \checkmark$$

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PRINCIPLE OF LINEAR SUPERPOSITION

We say that a system is linear if it obeys the "principle of linear superposition"

Take a function $f(x)$. If

$$\begin{cases} \underline{f(x_1 + x_2 + \cdots)} = \underline{f(x_1) + f(x_2)} + \cdots & \text{"additivity"} \\ \text{and} & \\ \underline{f(\alpha x_1)} = \underline{\alpha f(x_1)} & \text{"scaling"} \quad \alpha \text{ is a constant} \end{cases}$$

Then $f(x)$ is a linear process.

ex) Let $f = e^x$

$$f(x_1+x_2) = e^{x_1+x_2}$$

$$f(x_1)+f(x_2) = e^{x_1}+e^{x_2}$$

$$e^{x_1+x_2} \neq e^{x_1}+e^{x_2}$$ additivity does not hold.

Test scaling

$$f(\alpha x_1) = e^{\alpha x_1}$$ α a constant

$$\alpha f(x_1) = \alpha e^{x_1}$$ not equal either

fails scaling

$\therefore$ $f(x)=e^x$ is not linear //

ex) $f(x) = \alpha x$ (α is a constant) is linear, since

$$f(x_1+x_2) = \alpha x_1 + \alpha x_2 = f(x_1)+f(x_2)$$

and

$$f(a\alpha x) = a\alpha x = a f(\alpha x)$$ //

Go back to solutions to linear second order ode's. Focus on homogeneous problem:

$$L\,y(t) = 0$$

L is 2nd order linear operator.

The general solution is

$$y = c_1 y_1(t) + c_2 y_2(t)$$

c_1 & c_2 are constants.

Where $L y_1 = 0$

and $L y_2 = 0$

Rmk:

① Any solution to $L y(t) = 0$

can be expressed as a linear combination of functions y_1 & y_2, each satisfying

$$Ly_i(t) = 0 \quad i = 1, 2.$$

(2) Since c_1 & c_2 are arbitrary constants the solution is a "2-parameter family of solutions."

How to find c_1 & c_2? Via I.C.

(3) The solution of $Ly = 0$ has "2 degrees of freedom": these are $y_1(t)$ and $y_2(t)$ //

EXISTENCE & UNIQUENESS OF IVP SOLUTIONS VIA THE WRONSKIAN:

We can use the Wronskian to find whether solution exists & is unique

Consider IVP $\begin{cases} Ly(t) = 0 & \text{ODE} \\ y(t_0) = Y_0 \quad y'(t_0) = Y_1 & \text{I.C.} \end{cases}$

provided L has certain properties ODE, will have a unique solution for $t \in (\alpha, \beta)$. To see this, consider the following:

It's clear that for $y = c_1 y_1(t) + c_2 y_2(t)$

$$L(c_1 y_1 + c_2 y_2) = c_1 L y_1 + c_2 L y_2 = 0$$

since $Ly_1 = 0$ & $Ly_2 = 0$. The c_1 & c_2 are determined by applying I.C; and then

$(\mathcal{Y})$ $\quad y = c_1 y_1 + c_2 y_2$ will then have specific values of c_1 & c_2.

Apply I.C. using $(\mathcal{Y})$

$$(\star)\begin{cases} y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = Y_0 \\ y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = Y_1 \end{cases}$$

We can write down $(\star)$ as a matrix product:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$$

Solve the simultaneous equations for c_1 and c_2

Use Cramer's Rule:

$$c_1 = \frac{1}{W} \det \begin{bmatrix} Y_0 & y_2(t_0) \\ Y_1 & y_2'(t_0) \end{bmatrix}$$

$$W = \det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$$

W is the "Wronskian"

$$W = y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)$$

$$c_2 = \frac{1}{W} \det \begin{bmatrix} y_1(t_0) & Y_0 \\ y_1'(t_0) & Y_1 \end{bmatrix}$$

Rmk: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the $\det A = ad - bc$

Rmk: Since $W = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)$ is in the denominator of c_1 & c_2, then $W \neq 0$ for c_1 & c_2 to be defined.

Th'm The IVP $\begin{cases} Ly = 0 \\ y(t_0) = Y_0 \quad y'(t_0) = Y_1 \end{cases}$

has a unique solution

$$y = c_1 y_1(t) + c_2 y_2(t),$$

where $Ly_i = 0 \qquad i = 1, 2,$

iff $W = y_1 y_2' - y_1' y_2$ is **NOT ZERO** at $t = t_0$

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ex) $y'' + \omega^2 y = 0$ ω is a ≥ 0 constant

has fundamental solutions

$$y_1 = \cos(\omega x) \qquad y_2 = \sin(\omega x)$$

$$\therefore \; y(x) = C_1 \cos(\omega x) + C_2 \sin(\omega x)$$

Check: if $L = \frac{d^2}{dx^2} + \omega^2$

$$Ly_1 = 0 \quad \& \quad Ly_2 = 0$$

Let compute the Wronskian:

$$W = y_1 y_2' - y_2 y_1'$$

$y_1 = \cos\omega x$ $y_1' = -\omega \sin\omega x$

$y_2 = \sin\omega x$ $y_2' = \omega \cos\omega x$

$$W = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$= \cos\omega x \, \omega\cos\omega x - \sin\omega x(-\omega\sin\omega x)$$

$$= \omega\cos^2\omega x + \omega\sin^2\omega x$$

$$= \omega(\cos^2\omega x + \sin^2\omega x) = \omega$$

W is never zero (unless $\omega = 0$)

$\therefore$ the solution is unique is valid for any $t \in (-\infty, \infty)$

ex) $y'' + \omega^2 y = 0$

$y(0) = Y_0 \quad y'(0) = Y_1$

Can also determine that solution exists & is unique for $t \in (-\infty, \infty)$

ex) Find the largest interval in t for which a solution exists & is unique, to

$$\text{IVP} \begin{cases} y'' + \dfrac{\ln t}{t^2-2^2} y' + \dfrac{t}{t^2-2^2} y = \dfrac{\sin t}{t^2-2^2} \\ y(1) = 6 \qquad y'(1) = -2 \end{cases}$$

the has singularities at $t = \pm 2$

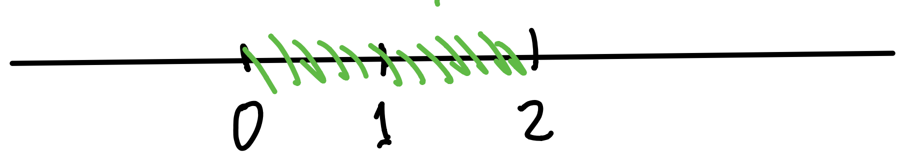
and is undefined for $t \leq 0$

$$\therefore \quad t \in (\alpha, \beta) \in (0, \infty) \setminus 2$$

as far as the ODE is concerned.

Add I.C. to analysis:

$t \in (0, 2)$



t_0 (when I.C. are defined)

ex) Determine whether $y_1 = e^{2t}$ and $y_2 = 6e^{2t}$ are fundamental solutions to

$$y'' - 4y = 0$$

$(*)$ $Ly_1 = 0$ $\quad y_1' = 2e^{2t}$ $\quad y_1'' = 4e^{2t}$

Substitute into $(*)$

$$4e^{2t} - 4e^{2t} = 0 \checkmark$$

Try $Ly_2 = 0$

$$6(4e^{2t} - 4e^{2t}) = 0 \checkmark$$

Let confirm we don't have a solution

$$y = c_1 \hat{y}_1(t) + c_2 \hat{y}_2(t)$$

with $\hat{y}_1$ and $\hat{y}_2$ satisfying $L\hat{y}_i = 0$

and $\hat{y}_1$ & $\hat{y}_2$

not equal to each other (to within a constant)

To confirm: compute $W(e^{2t}, 6e^{2t})$

$$y_1 = e^{2t} \qquad y_1' = 2e^{2t}$$

$$y_2 = 6e^{2t} \qquad y_2' = 12e^{2t}$$

$$W(e^{2t}, 6e^{2t}) = \det \begin{vmatrix} e^{2t} & 2e^{2t} \\ 6e^{2t} & 12e^{2t} \end{vmatrix}$$

$$= 12e^{2t}e^{2t} - 6e^{2t}2e^{2t} = 0$$

$\therefore$ W is 0 for all t!

$\therefore$ $y = c_1 e^{2t} + c_2 6e^{2t}$ is not a unique & general solution to

$$y'' - 4y = 0$$

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ABEL'S THM: Suppose $Ly=0$ where $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$; p, q are continuous on some open interval I.

Can we infer whether $Ly=0$ will generate a fundamental solution set, without first knowing y_1 & y_2?

Yes, via Abel's theorem: it says that the Wronskian

$$W(y_1,y_2)(t) = c \exp\left[-\int p(t)\,dt\right]$$

where c is a constant not dependent on t.

Further $W=0$ iff $c=0$, or $W \neq 0$ if $c \neq 0$, regardless of $p(t)$ in I //

Pf: let y_1 & y_2 be solutions to

$$L y_i = 0 \qquad i = 1, 2$$

$\therefore$

$$y_1'' + p y_1' + q y_1 = 0 \qquad (A)$$

$$y_2'' + p y_2' + q y_2 = 0 \qquad (B)$$

Multiply (A) by $-y_2$ & (B) by y_1

Then add the 2 resulting equations:

$$-y_2 y_1'' - p y_2 y_1' \cancel{- q y_2 y_1} = 0$$

$$y_1 y_2'' + p y_1 y_2' \cancel{+ q y_1 y_2} = 0$$

$$\underbrace{y_1 y_2'' - y_2 y_1''}_{\frac{dW}{dt}} + p\underbrace{(y_1 y_2' - y_2 y_1')}_{W} = 0 \qquad (\ddagger)$$

if $W = y_1 y_2' - y_2 y_1'$

$$\frac{dW}{dt} = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

$\therefore$ (H) $\quad \frac{dW}{dt} + pW = 0$ or

$$\frac{dW}{W} = -p\,dt$$

integrate b.s.

$$\ln|W| = -\int^t p(s)\,dt + k$$

$$\therefore W = c\,e^{-\int p\,dt}$$

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ex) Find the largest interval for which a fundamental solution exists, to

(✱) $y'' - 5y' + 6y = 0$

Use Abel's theorem:

$$W = ce^{-\int p\,dt}$$

here $p = -5$

$$W = ce^{+5\int dt} = ce^{+5t}$$

which can never be 0 (for any t) unless $c = 0$

$\therefore$ $I = (-\infty, \infty)$ ✱ has a fundamental solution.

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ex) Find the Wronskian to

$$x^2y'' - 2xy' - 4y = 0 \quad (\$)$$

$$y'' - \frac{2x}{x^2}y' - \frac{4}{x^2}y = 0$$

Here $p(x) = -\frac{2}{x}$

$$W = c\, e^{+\int \frac{2}{x}dx} = c e^{2\int \frac{dx}{x}} = ce^{2\ln x}$$

$$W = cx^2$$

so $(-\infty, 0)$ or $(0, \infty)$ are reasonable intervals.