2nd ORDER LINEAR ODES

3.1 Equations with Constant coefficients

Euler's Equation

The general 2nd order ODE

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

is linear, if it can be written as

$$(\ast) \quad a_2(t)\,y'' + a_1(t)\,y' + a_0(t)\,y = h(t)$$

$a_2(t) \neq 0$; a_1 or a_0 can be zero

$a_i(t)$ are functions of t (known)

$h(t)$ is function of t (given).

If all $a_i(t) = a_i$, constants

$(\star)$ is a linear CONSTANT COEFFICIENTS SECOND ORDER (NON HOMOGENEOUS) ODE:

$$ay'' + by' + cy = h(t) \quad \text{(C.C.)}$$

$a \neq 0$; a, b, c are constants.

Another linear 2nd order ODE:

SECOND ORDER EULER'S EQUATION (NON HOMOGENEOUS) ODE:

$$at^2 y'' + bty' + cy = g(t) \quad \text{(E.E.)}$$

a, b, c are constants

def: let L denote a **linear differential operator**,

so that (E.E.) can be written compactly as

$Ly = g(t)$, where $L = at^2 \frac{d^2}{dt^2} + bt\frac{d}{dt} + c$.

For the (C.C.) we can write $Ly = g(t)$

but in this case, $L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c$.

// Consider the 2nd order linear ODE:

$$(\ddagger) \qquad Ly = f(t)$$

see (*) for L, $f(t)$ is known.

the general solution of $(\ddagger)$ is $y = y_H + y_p$

y_H is homogeneous solution

y_p is particular solution

Where y_H solves

$$Ly_H = 0$$, and y_p solves

$$Ly_p = f(t).$$ Note that

$$Ly = \underbrace{Ly_H}_{=0} + Ly_p = f(t)$$ is satisfied, since

and $Ly_p = f(t)$.

A few facts about $(\dagger)$:

(1) The equation $Ly_H = 0$

always has a solution $y_H = 0$, known

as the the trivial solution.

(2) if y_1 is a solution to $Ly_1 = 0$

then αy_1 is also a solution to

$Ly = 0$, α is a constant.

(3) if y_1 & y_2 are solutions

$Ly_1 = 0$ and $Ly_2 = 0$,

then $y = \alpha y_1 + \beta y_2$

is a solution to $Ly = 0$

here α, β are constants.

Focus on Solving C.C. (Homogeneous Case)

$$(*) \quad \begin{cases} Ly = 0, \text{ where} \\ L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c \end{cases}$$

$a \neq 0$; a, b, c constants.

Guess $y = Be^{mt}$ (A)

B, m are constants.

Replace Guess into (*):

Need $\frac{dy}{dt} = Bme^{mt}$ (B)

$\frac{d^2y}{dt^2} = Bm^2e^{mt}$ (C)

Replace (A), (B), (C) into (*):

$$\underbrace{a\,Bm^2e^{mt}}_{a\frac{d^2y}{dt^2}} + \underbrace{bBme^{mt}}_{b\frac{dy}{dt}} + \underbrace{cBe^{mt}}_{cy} = 0$$

factor out Be^{mt}:

$$Be^{mt}\left(am^2 + bm + c\right) = 0$$

Since $B \neq 0$ & $e^{mt} \neq 0$
for all t

$\therefore$ $am^2 + bm + c = 0$
Characteristic Equation

$\therefore$ the roots of
the characteristic equation determine
the values of m: m_1, m_2 in solution:

$$m_{1,2} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \quad (\#)$$

$\therefore$ the solution of

$$Ly = 0 \quad \text{in } (\dot{\not{*}})$$

$$y = \alpha e^{m_1 t} + \beta e^{m_2 t}$$

α, β are constants, arbitrary.

we call this the

general solution to ($\not{F}$).

Note: there are always two roots m_1, m_2:

Example Characteristic equation ($\#$)

Roots are real and distinct: if the discriminant

$$\sqrt{b^2-4ac} \text{ has } b^2 > 4ac.$$

Roots are complex and complex conjugates:

$$m_{1,2} = \delta \pm i\varepsilon \qquad i = \sqrt{-1}$$

$$\delta = -\frac{b}{2a}, \; \varepsilon = \frac{1}{2a}\sqrt{4ac-b^2}$$

$$\text{where } 4ac > b^2$$

Roots are real and equal:

$$m_{1,2} = m = -\frac{b}{2a}, \text{ the discriminant} = 0:$$

i.e. when $b^2 = 4ac$

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Consider E.E. homogeneous case

$$a t^2 \frac{d^2y}{dt^2} + b t \frac{dy}{dt} + c y = 0$$

or $\quad L y = 0$

Guess $\quad y = B t^m \quad$ (A)

need: $\quad \frac{dy}{dt} = B m t^{m-1} \quad$ (B)

$$\frac{d^2y}{dt^2} = B m(m-1) t^{m-2} \quad \text{(C)}$$

substitute

A, B, C into (E.E.):

$$at^{2}\,Bm(m-1)t^{m-2} + bt^{1}\,Bmt^{m-1} + cBt^{m} = 0$$

$$Bt^{m}\left[a\,m(m-1) + bm + c\right] = 0$$

The characteristic equation for E.E:

$$am^{2} - a + bm + c = 0$$

or $$am^{2} + (b-a)m + c = 0$$

or $$m^{2} + \frac{b-a}{a}m + \frac{c}{a} = 0$$

So 3 possible types of roots $m_{1,2}$

2 real & distinct if $\left(\frac{b-a}{a}\right)^{2} > \frac{4a}{c}$

1 real root (repeated) $\left(\frac{b-a}{a}\right)^{2} = 4a/c$

2 complex conjugate roots when

$$\left(\frac{b-a}{a}\right)^2 < \frac{4a}{c}$$

The general solution to (E.E.)

$$y = \alpha t^{m_1} + \beta t^{m_2}$$

$m_{1,2}$ are the roots of characteristic equation, α, β are constants.

ex) $y'' - 3y' + 2y = 0$ (ODE)

C.C.: Assume $y = Be^{mt}$ substitute onto ODE

$$(m^2 - 3m + 2)e^{mt} = 0$$

so the characteristic equation is

$$m^2 - 3m + 2 = 0$$

Factorize, to find roots:

$$(m-1)(m-2)=0$$

$$\therefore \quad m_1=1, \quad m_2=2$$

$$y=\alpha e^{t}+\beta e^{2t}$$

ex) $t^2y''-2ty'-4y=0$

is E.E. $\therefore$ assume a solution of the form

$$y=t^m$$

$$t^2 m(m-1)t^{m-2}-2t\,m t^{m-1}-4t^m=0$$

$$t^m\left[m^2-m-2m-4\right]=0$$

Hence the characteristic equation is

$m^2 - 3m - 4 = 0$, factoring ,

or $(m+1)(m-4) = 0$ $m_1 = -1$

$m_2 = 4$

$\therefore y = \alpha t^{-1} + \beta t^4$