2.6 Numerical Approximations to IVP

Consider the first order ODE and I.C

$$\begin{cases} \dfrac{dy}{dt} = f(t,y) & t>0 \\ y(0) = y_0 \end{cases}$$

If an analytical solution cannot be found we need to resort to numerical means, or to Taylor series approximations.

We consider here the simplest numerical method, called "EXPLICIT EULER". It is neither always applicable (in some IVP it yields nonsense) and it is not the most efficient (it might require enormous computer resources) and thus other alternative methods are available.

let $t_i = i\Delta t$ be discrete time, Δt is a fixed time step, $i = 0, 1, 2 \ldots$

Explicit Euler

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) & t > 0 \\ y(0) = y_0 \end{cases}$$

$$\left.\frac{dy}{dt}\right|_{t=t_i} \approx \frac{y_{i+1} - y_i}{\Delta t_i}$$

so at $t = t_i$

$$\frac{y_{i+1} - y_i}{\Delta t} = f(t_i, y_i) \quad , \text{with } y(t_0 = 0) = y_0$$

or

"Explicit Euler Scheme"

$$\begin{cases} y_{i+1} = y_i + \Delta t\, f(t_i, y_i) & i = 0, 1, 2 \ldots \\ \text{at } i = 0 \quad y_i = y_0 \end{cases}$$

Which leads to

y_0 known

$$y_1 = y_0 + \Delta t\, f(0, y_0)$$

$$y_2 = y_1 + \Delta t\, f(\Delta t, y_1)$$

$$y_3 = y_2 + \Delta t\, f(2\Delta t, y_2)$$

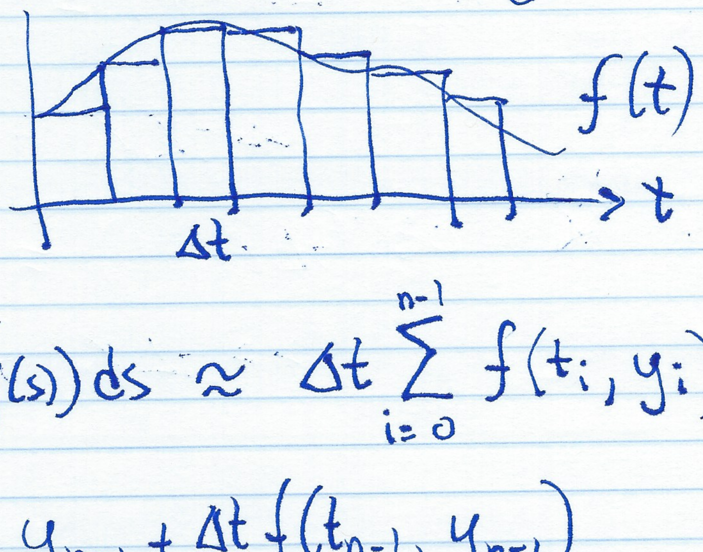
$$\vdots$$

$$y_n = y_{n-1} + \Delta t\, f(t_{n-1}, y_{n-1}) \; ; \quad t_n = n\Delta t$$

$$n = \cancel{0}1, 2, 3 \ldots$$

Which is equivalent to the left hand rule of approximate integration. To see this, write

$y' = f(t, y)$ as $y(t) = \displaystyle\int_0^t f(s, y(s))\,ds + y_0$



$$\int_0^t f(s, y(s))\,ds \approx \Delta t \sum_{i=0}^{n-1} f(t_i, y_i) = \Delta t \sum_{i=0}^{n-1} f_i$$

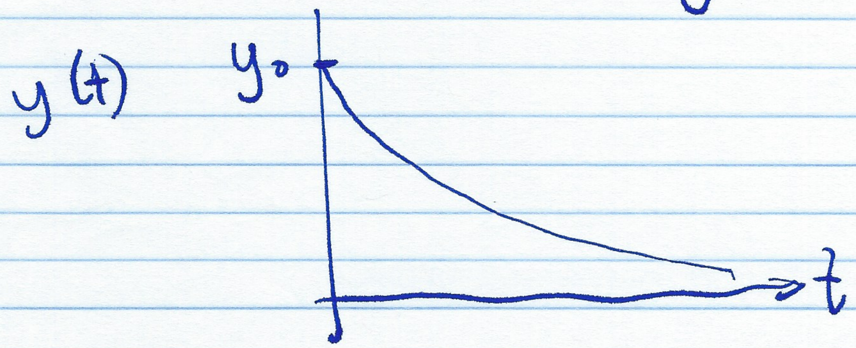
or $y_n = y_{n-1} + \Delta t\, f(t_{n-1}, y_{n-1})$

EXPLICIT EULER, APPLIED TO A SIMPLE PROBLEM

Consider the following initial value problem:

$$\text{IVP} \begin{cases} \dfrac{dy}{dt} = -\lambda y, & t > 0 \quad \lambda > 0, \text{ constant} \\ y(0) = y_0 \end{cases}$$

The exact solution is $y(t) = y_0 e^{-\lambda t}$ $t > 0$



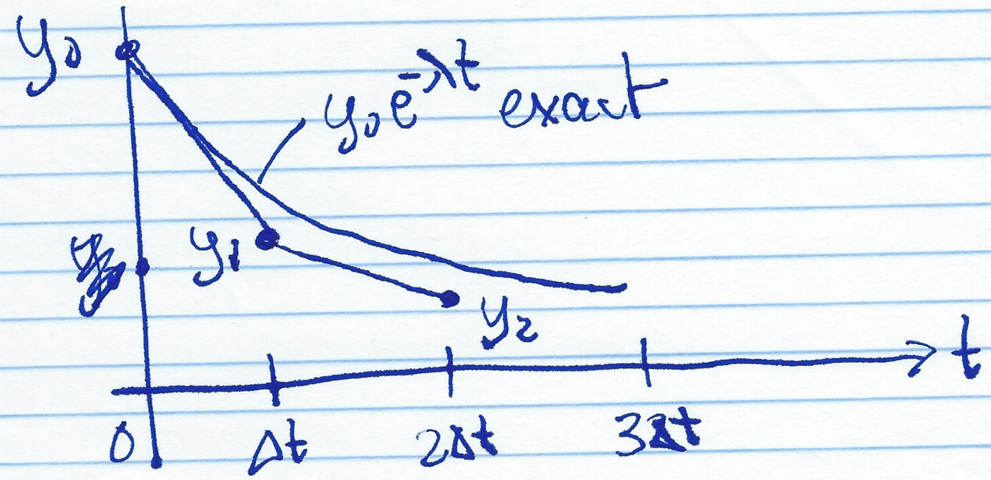
The IVP states that the slope of the solution is $-\lambda y$.

Let $t_j = 0, \Delta t, 2\Delta t, \dots$, where $\Delta t < 1$

$$t_j = j\Delta t \qquad j = 0, 1, 2 \dots$$

is "discrete time", Δt is the "time step"

Δt need not be constant, but we'll make it that way here



$$\frac{dy}{dt} \approx \frac{y_{i+1} - y_i}{\Delta t}$$

$$\begin{cases} \dfrac{dy}{dt} = -\lambda y \\ y(0) = 0 \end{cases}$$ is approximated as

$$\begin{cases} \dfrac{y_{i+1} - y_i}{\Delta t} = -\lambda y_i & i = 1, 2 \dots \\ y_0 \text{ known.} \end{cases}$$

$$\begin{cases} y_{i+1} = y_i - \Delta t \lambda y_i = (1 - \lambda \Delta t) y_i & i = 1, 2 \dots \\ y_0 \text{ known} \end{cases}$$

So y_0 is known $\qquad i=0$

"The difference equations"

$$y_1 = (1-\lambda\Delta t)y_0 \qquad i=1$$

$$y_2 = (1-\lambda\Delta t)y_1 = (1-\lambda\Delta t)^2 y_0 \qquad i=2$$

$$\vdots$$

$$(\star) \quad y_n = (1-\lambda\Delta t)^n y_0 \qquad i=n$$

is an approximation to

$$(\$) \quad y(t_n) = e^{-\lambda t_n} y_0 \quad \text{(the exact solution)}$$

Taylor series $e^{-\lambda t_n} = e^{-\lambda \Delta t n} = (e^{-\lambda\Delta t})^n$

$$\approx \left(1 - \lambda\Delta t + \tfrac{1}{2}\lambda^2\Delta t^2 - \dots\right)^n$$

if $\lambda\Delta t \ll 1$

$\approx (1-\lambda\Delta t)^n$, so $(\star)$ approximates

$(\$)$ if $\lambda\Delta t \ll 1$

This is a very special case because we can solve the general finite difference equations.

The Error in the approximation:

the absolute error is

$$E(t_n) = |\, y(t_n) - y_n \,|$$

exact solution at $t = t_n$

given by explicit Euler

$E(t_n) = c\,\Delta t$, where c is some constant

What this means is that as $\Delta t \to 0$ $E(t_n)$ also goes to 0, so solution is more "accurate".

If you want double the accuracy, you'd need to reduce the step size by $\frac{1}{2}$. If you'd want four times more accuracy, you'd reduce the step size by $\frac{1}{4}$.

Generally, this means that every time you decrease the step size by 2, you'd double

• the time it takes to compute an approximate answer.

Notes: (1) There are more efficient methods than explicit Euler Method when ~~doubling~~ halving the time step leads to $E(t_n) = c\Delta t^p$ where $p > 1$.

(2) Not every IVP can be approximated by the Explicit Euler method, even if Δt is made really tiny.

(3) Δt cannot be made arbitrarily large on many IVP. The method will go unstable and compute nonsense (not just inaccurate) outcomes.