2.6 EXACT DIFFERENTIALS & EXACT 1st order ODE's:

let $(\yen)$ $\psi(x,y) = C$,

C is constant

$\psi(x,y)$ is continuous & differentiable with respect to x, y.

Differentiate $(\yen)$:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad (\$)$$

So, suppose we are given $(\$)$ then the solution to $(\$)$ is $(\yen)$.

So the 1st order differential equation

$$d\psi = 0$$

where $d\psi$ is given by ($) has a solution given by (*).

If given $M(x,y)\,dx + N(x,y)\,dy = 0$, this equation would be $d\psi = 0$ if $M = \frac{\partial \psi}{\partial x}$ and $N = \frac{\partial \psi}{\partial y}$

That is:

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$= M(x,y)\,dx + N(x,y)\,dy$$

here

$$M(x,y) = \frac{\partial \psi}{\partial x} \quad \text{A}$$

$$N(x,y) = \frac{\partial \psi}{\partial y} \quad \text{B}$$

Special Case: Suppose

$M(x,y) = M(x)$

$N(x,y) = N(y)$

$M(x)\,dx = -N(y)\,dy$

$\therefore$ separable

GENERAL CASE:

$d\psi = M(x,y)\,dx + N(x,y)\,dy = 0$

iff

$M = \frac{\partial \psi}{\partial x}$ (A)

$N = \frac{\partial \psi}{\partial y}$ (B)

AND $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (★)

Condition follows from cross-differentiating

(A) & (B):

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$$

Conditions (A), (B) and (✱) identifies a 1st order equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = d\psi$$

Hence, it has a solution (✱).

ex) IVP $\begin{cases} 2x - 5y + (3y^2 - 5x)\dfrac{dy}{dx} = 0 & \text{ODE} \\ y(0) = 1 & \text{I.C.} \end{cases}$

(1) Put ODE in form $M\,dx + N\,dy = 0$

$$(2x-5y)\,dx+(3y^2-5x)\,dy=0 \quad (*)$$

② Identify M, N:

let $M(x,y)=2x-5y$

$N(x,y)=3y^2-5x$

③ Test to see if $(*)$ is exact:

$$\frac{\partial M}{\partial y}=-5$$

$$\frac{\partial N}{\partial x}=-5$$ Yes, condition $(**)$ is satisfied:

See that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \quad \therefore (*) \text{ is exact.}$$

$M\,dx + N\,dy = 0$ is really

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0,$$

$\therefore\ \psi = c$ is the solution.

$$(**)\quad \frac{\partial \psi}{\partial y} = N(x,y) = 3y^2 - 5x$$ from (*).

Integrate (either N, M)

integrate (**) wrt y:

$$\psi(x,y) = y^3 - 5xy + h(x) \quad (***)$$

But $\frac{\partial \psi}{\partial x} = M = 2x - 5y$ from (*).

Differentiate (***) wrt x. The result must equal M:

$$\frac{\partial \psi}{\partial x} = -5y + \frac{dh}{dx} = M = 2x - 5y$$

$$\therefore \quad \frac{dh}{dx} = 2x$$ integrating:

$$h = x^2 + c$$

Substitute $h(x)$ into (***) to get solution:

$$\psi = y^3 - 5xy + x^2 + c = \tilde{c}$$

$$\text{or } y^3 - 5xy + x^2 = C$$

APPLY IC, if given:

from IVP: $y(0) = 1$ in this problem.

$$\therefore \quad C = 1$$

Solution IVP $y^3 - 5xy + x^2 = 1$ //

ex) $x\frac{dy}{dx} = 2xe^x - y + 6x^2$

$x\,dy - (2xe^x - y + 6x^2)\,dx = 0$

$M = -(2xe^x - y + 6x^2)$

$N = x$

is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. is ODE exact?

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 1$

$\Rightarrow$ EXACT!

$\therefore$

$$x\,dy - (2xe^{x} - y + 6x^2)\,dx = 0$$

$$\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = 0$$

$$\frac{\partial \psi}{\partial y} = x \implies \text{integrate wrt } y:$$

$$\psi(x,y) = xy + h(x)$$

Differentiate above wrt x:

$$\frac{\partial \psi}{\partial x} = y + h'(x) = -2xe^{x} + y - 6x^2$$

$$\therefore \frac{dh}{dx} = -2xe^{x} - 6x^2$$

integrate above wrt x:

$$h(x) = -2xe^{x} - 2e^{x} - 2x^3 + \tilde{c}$$

$$\therefore \Psi = xy - 2xe^x - 2e^x - 2x^3 + \hat{c}, \text{ or}$$

$$xy - (2x+2)e^x - 2x^3 = C$$

Integrating Factors

Sometimes we can turn a 1st order ODE into an exact 1st order ODE (if it's not already):

$$\text{Given } \hat{M}(x,y)\,dx + \hat{N}(x,y)\,dy = 0 \quad (\ddagger)$$

and found that $\partial \hat{M}/\partial y \neq \partial \hat{N}/\partial x$ $\therefore$ NOT EXACT.

We can try using INTEGRATING FACTOR:

Multiply $(\ddagger)$ by $\mu(x,y)$ integrating factor

$$(\$) \quad \underbrace{\mu(x,y)\hat{M}}_{M}\,dx + \underbrace{\mu(x,y)\hat{N}}_{N}\,dy = 0$$

let

so that $M\,dx + N\,dy = 0$ ($)

is $d\psi = 0$

hence $\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$

is, by comparing to (†)

$$\frac{\partial \psi}{\partial x} = \mu \tilde{M}$$

so

$$\frac{\partial \psi}{\partial y} = \mu \tilde{N}$$

know that ($) is exact iff

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\mu \tilde{M}\right)$$

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial x}(\mu \tilde{N})$$ are equal:

$$\therefore \quad \frac{\partial}{\partial y}(\mu\tilde{M}) = \frac{\partial}{\partial x}(\mu\tilde{N}).$$

Expanding this

$$\cancel{(*)} \quad \frac{\partial\mu}{\partial y}\tilde{M} + \mu\frac{\partial\tilde{M}}{\partial y} = \frac{\partial}{\partial x}\mu\,\tilde{N} + \mu\frac{\partial\tilde{N}}{\partial x}$$

$$(*) \quad \frac{\partial\mu}{\partial y}\tilde{M} - \frac{\partial\mu}{\partial x}\tilde{N} = \mu\left(\frac{\partial\tilde{N}}{\partial x} - \frac{\partial\tilde{M}}{\partial y}\right)$$

NEED TO FIND μ SATISFYING (*). How?

Try this: suppose that $\mu(x,y) = \mu(x)$

So $\cancel{\ }$ is 0 and (*) is then

$$-\frac{\partial\mu}{\partial x}\tilde{N} = \mu\left(\frac{\partial\tilde{N}}{\partial x} - \frac{\partial\tilde{M}}{\partial y}\right)$$

$$\text{or} \quad -\frac{\partial \mu}{\mu} = \frac{1}{\tilde{N}}\left(\frac{\partial \tilde{N}}{\partial x} - \frac{\partial \tilde{M}}{\partial y}\right)\partial x \qquad (¥)$$

This is a ODE for $\mu = \mu(x)$ which we can (hopefully) solve.

If a solution for μ is found then

Multiply (‡) by μ. The resulting ODE is exact then apply exact ODE recipe for solution.

If $\mu = \mu(x)$ did not work out, try $\mu = \mu(y)$

$$(\ast) \quad \frac{\partial \mu}{\partial y}\tilde{M} + \mu\frac{\partial \tilde{M}}{\partial y} = \frac{\partial}{\partial x}\mu\,\tilde{N} + \mu\frac{\partial \tilde{N}}{\partial x}$$

// drops out since $\mu = \mu(y)$

$$\text{then} \quad \frac{\partial \mu}{\partial y}\tilde{M} + \mu\frac{\partial \tilde{M}}{\partial y} = \mu\frac{\partial \tilde{N}}{\partial x}$$

$$\frac{\partial \mu}{\partial y} = \frac{\mu}{\tilde{M}}\left(\frac{\partial \tilde{N}}{\partial x} - \frac{\partial \tilde{M}}{\partial y}\right)$$

or

$$\frac{\partial \mu}{\mu} = \frac{1}{\tilde{M}}\left(\frac{\partial \tilde{N}}{\partial x} - \frac{\partial \tilde{M}}{\partial y}\right) dy$$

Solve for $\mu = \mu(y)$. Multiply (†) by μ then resulting ODE is exact. Follow usual procedure.

ex) $\frac{dy}{dx} = e^{2x} + y - 1$ or

$$\underbrace{1}_{\tilde{N}} dy - \underbrace{(e^{2x} + y - 1)}_{\tilde{M}} dx = 0$$

$\tilde{N} = 1$ $\qquad$ $\tilde{M} = -(e^{2x} + y - 1)$

Clearly, $\frac{\partial \tilde{M}}{\partial y} \neq \frac{\partial \tilde{N}}{\partial x}$

so ODE is not exact.

Introduce an integrating

Try $\mu = \mu(x)$ $\quad -\frac{\partial\mu}{\mu} = \frac{1}{\hat{N}}\left(\frac{\partial\hat{N}}{\partial x} - \frac{\partial\hat{M}}{\partial y}\right)dx$

$$\mu(x)\,dy - \mu(x)(e^{2x}+y-1)\,dx = 0$$

or $\frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx = 0$

(if exact). Using $(\ast)$

$$-\frac{\partial\mu}{\mu} = \frac{1}{\hat{N}}\left(\frac{\partial\hat{N}}{\partial x} - \frac{\partial\hat{M}}{\partial y}\right)dx$$

where $\hat{N} = 1$ and $\hat{M} = -(e^{2x}+y-1)$ we get

$$-\frac{1}{\mu}\frac{d\mu}{dx} = -\frac{\partial}{\partial y}(-e^{2x}+y-1) = -1$$

so $\frac{1}{\mu}\frac{d\mu}{dx} = -1$

or $\frac{d\mu}{\mu} = -dx$

Solving $\ln|\mu| = -x + c$

(forget constant)

$\therefore \ \mu = e^{-x}$

Multiply ODE by μ. Result is an exact equation:

$$(3) \quad \underbrace{e^{-x}}_{N} dy - \underbrace{e^{-x}(e^{2x}+y-1)}_{M} dx = 0$$

$$N = e^{-x}$$

$$M = -e^{-x}\left(e^{2x}+y-1\right)$$

Check for exact:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$-e^{-x} = -e^{-x} \qquad \checkmark$$

Solve (7):

$$e^{-x}\,dy - \underline{e^{-x}(e^{2x}+y-1)}\,dx = 0$$

$$\frac{\partial \psi}{\partial y}\,dy + \frac{\partial \psi}{\partial x}\,dx = 0$$

$\frac{\partial \psi}{\partial y} = e^{-x}$ integrate wrt

$$\psi = -e^{-x}y + h(x)$$

$$\therefore \frac{\partial \psi}{\partial x} = e^{-x}y + h'(x) = \underline{-e^{-x}(e^{2x}+y-1)}$$

$$-\cancel{e^{-x}y} + h'(x) = -e^{x} - \cancel{e^{-x}y} + e^{-x}$$

$h'(x) = e^{-x} - e^{x}$ integrate:

$$h = -e^{-x} - e^{x} + c$$

$$\therefore \psi = -e^{-x}y - e^{-x} - e^{x} + c \quad //$$

Bernoulli Equation

(BE) $\dfrac{dy}{dx} + Q(x)y = f(x)y^n \qquad n \in \mathbb{R}$

we know how to solve for $n = 1$ (linear first order ODE. For $n \neq 1$, we use a trick:

Trick let $w = y^{1-n}$ (A)

$$dw = (1-n)y^{1-n-1}dy$$

solving for: $dy = \dfrac{1}{1-n} y^n dw.$ (A)

substitute (A) into (BE):

$$\frac{1}{1-n} y^n \frac{dw}{dx} + Q(x)y = f(x)y^n$$

$$\frac{1}{1-n} \frac{dw}{dx} + Q(x)\frac{y}{y^n} = f(x)$$

$$\frac{1}{1-n} \frac{dw}{dx} + Q(x)\underbrace{y^{1-n}}_{w} = f(x)$$

$$\frac{1}{1-n}\frac{dw}{dx} + Q(x)\,w = f(x)$$

$$\frac{dw}{dx} + (1-n)Q(x)\,w = (1-n)f(x)$$

1 order linear ODE for w which we know how to solve.

ex) $\dfrac{dy}{dx} - y = e^{x} y^{2 = n}$ (BE)

$$w = y^{1-n} = y^{1-2} = y^{-1}$$

then $dw = -y^{-2}\,dy \Rightarrow -y^{2}\,dw = dy$

$$-y^{2}\frac{dw}{dx} - y = e^{x} y^{2}$$

divide by y^{2}

$$-\frac{dw}{dx} - w = e^{x}$$

(✱) $\frac{dw}{dx} + w = -e^{x}$. 1st order linear ODE

let $I = e^{x}$, multiply ✱ by I

$$I\frac{dw}{dx} + wI = -e^{x}I$$

$$\frac{d}{dx}(Iw) = -e^{x}e^{x} = -e^{2x}$$

integrate

$$Iw = -\frac{1}{2}e^{2x} + c$$

$$w = \frac{1}{I}\left(-\frac{1}{2}e^{2x} + c\right)$$

$$w = -\frac{1}{2}\frac{e^{2x}}{e^{x}} + ce^{-x}$$

$$w = -\frac{1}{2}e^{x} + ce^{-x}$$

Substitute back $w = y^{1-2} = \frac{1}{y}$

$$\frac{1}{y} = -\frac{1}{2}e^{x} + ce^{-x}$$

or $y = \dfrac{1}{ce^{-x} - \frac{1}{2}e^{x}}$