# SECTION 2.5 DISCONTINUOUS FORCING

ex)

ODE

$$\text{IVP}\begin{cases} \dfrac{dy}{dx} + y = F(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 \le x \end{cases} \\ y(0) = 0 \end{cases}$$

I.C.

The trick is to enforce continuity of $y(x)$ for $x \geq 0$

The ODE solved using integrating factor:

$$(\bigstar)\quad y = e^{-x}\int^{x} e^{s} F(s)\, ds + e^{-x} c$$

For $0 \le x < 1$, let $y = y_I$

$$y_I = e^{-x}\int^{x} e^{s}\, 1\, ds + e^{-x} c$$

$y_I = e^{-x} e^{x} + c e^{-x}$. Next, find c:

for I.C. $y_I(0) = 0$ $\therefore$ $y_I = 1 - e^{-x}$ ($)

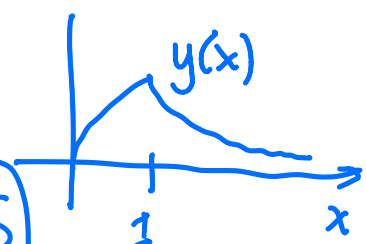
Now, for $0 \le x < 1$

For $x \ge 1$

★ $y_{II} = e^{-x} c_1$

We have $y = y_I$ for $0 \le x < 1$, and $y = y_{II}$ for $x \ge 1$. We require that $y_I(1) = y_{II}(1)$ & this determines $c_1$

Think of $y_I(1)$ as the initial conditions of $y_{II}$ at $x = 1$




$y_I(1) = 1 - e^{-1}$ using ($)

So $y_{II}(1) = 1 - e^{-1} = c_1 e^{-1}$ using (★)

$\therefore c_1 = e(1 - e^{-1})$ $\therefore$ $y(x) = \begin{cases} 1 - e^{-x} & 0 \le x < 1 \\ (e-1)e^{-x} & x > 1 \end{cases}$ //

## PARTIAL FRACTIONS

Take $f(x) = \frac{g(x)}{h(x)}$ where $h(x)$ is a product of polynomials. For certain forms of $h(x)$, one can write down $f(x)$ as a sum of ratios of functions, with simple denominators. We will consider, by example, how this works out for problems where $h(x)$ is a product of monomials

A monomial is a polynomial of first order

e.g. $x-\alpha$, where $\alpha$ is a constant, is a monomial

$(x-\alpha)^3$ is a product of 3 monomials.

**First step:** $f(x) = \frac{g(x)}{h(x)} = g(x) \cdot \frac{1}{h(x)}$

So work out the partial fraction of $1/h(x)$ and then multiply the partial fraction expansion by $g(x)$ to obtain $f(x)$.

**Second Step:** Pick a form for the partial fraction expansion that will work:

If $h(x) = \dfrac{1}{(x-a)^{m_1}(x-b)^{m_2}\cdots(\text{etc})}$ $\quad m_1, m_2 \geq 1$ integers

$$= \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_{m_1}}{(x-a)^{m_1}}$$

$$+ \frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \cdots\cdots + \frac{B_{m_2}}{(x-b)^{m_2}}$$

(etc)

**Third Step:** multiply both sides of $h(x)$ by $1/h(x)$, collect powers of $x$, then solve for all the constants $A_1, A_2 .. A_{m_1}, B_1, B_2 .. B_{m_2} \ldots$

ex) $f(x) = \dfrac{x^2}{(x-a)(x-b)} = \dfrac{g(x)}{h(x)} = x^2 \cdot \dfrac{1}{h(x)}$

Focus on $h(x)$:

$$h(x) = \frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

multiply both sides by $1/h(x)$:

$$1 = A(x-b) + B(x-a)$$

Collect powers of $x$:

$$\ldots 0 \cdot x^2 + 0 \cdot x^1 + 1x^0 = (-bA - aB)x^0 + (A+B)x^1 + 0 \cdot x^2 + \ldots$$

Match powers of $x$:

$$0 \cdot x^1 = (A+B)x^1$$

$$1 \cdot x^0 = (-bA - aB)x^0$$

$$\therefore \begin{cases} A + B = 0 & \Rightarrow B = -A \\ -bA - aB = 1 \end{cases}$$

$$\hookrightarrow -bA + aA = 1 \Rightarrow A = \frac{1}{a-b}$$

$$\therefore B = -\frac{1}{a-b}$$

$$\therefore\ h(x) = \frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right]$$

$$\therefore\ f(x) = \frac{1}{(a-b)}\frac{x^2}{(x-a)} - \frac{x^2}{(a-b)(x-b)}$$

//

ex) $$\frac{1}{(x-a)(x-b)^3} = \frac{A}{(x-a)} + \frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \frac{B_3}{(x-b)^3}$$

Multiply both sides by $(x-a)(x-b)^3$

$$1 = A(x-b)^3 + B_1(x-a)(x-b)^2 + B_2(x-a)(x-b) + B_3(x-a)$$

collect powers of $x$:

$$\ldots 0x^1 + 1\cdot x^0 = [Ab^3 - ab^2B_1 + abB_2 - aB_3]\,x^0$$
$$+ [3Ab^2 + b^2B_1 + 2aB_1 - aB_2 - bB_2 + B_3]\,x^1$$
$$+ [-3bA - 2B_1 - aB_1 + B_2]\,x^2$$
$$+ [A + B_1]\,x^3$$

Match coefficients of powers of $x$:

$$\star\quad\begin{cases} 1 = [A b^3 - a b^2 B_1 + a b B_2 - a B_3] \\ 0 = [3 A b^2 + b^2 B_1 + 2 a B_1 - a B_2 - b B_2 + B_3] \\ 0 = [-3 b A - 2 B_1 - a B_1 + B_2] \\ 0 = A + B_1 \end{cases}$$

Solve for $A, B_1, B_2, B_3$ using $\star$

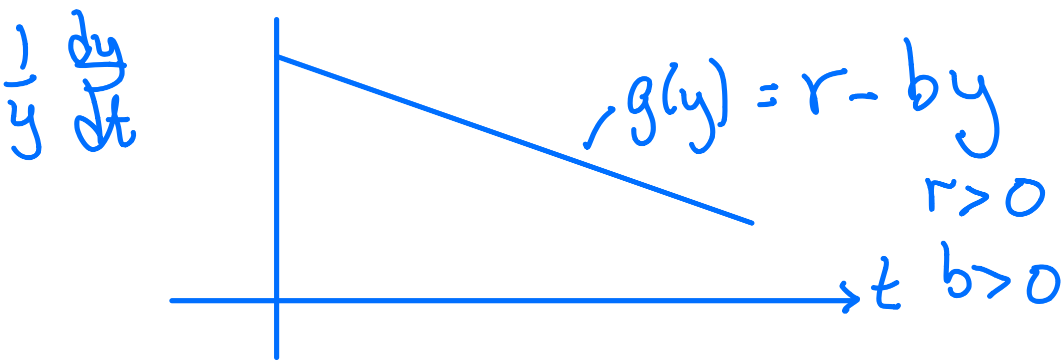
SEE OTHER FORMS OF $h(x)$ AMENABLE TO PARTIAL FRACTIONS ON THE WEB...

## POPULATION DYNAMICS

$\frac{dy}{dt} = r y$ the population grows/decay exponentially

$\frac{1}{y}\frac{dy}{dt} = r$ the relative rate is constant $r$

This model was OK for US population prior to WW II. But after, the model implied by data was given by




$$\frac{dy}{dt} = y\,g(y) = y(r - by) = f(y)$$

$$\frac{dy}{dt} = (r - by)y = f(y)$$

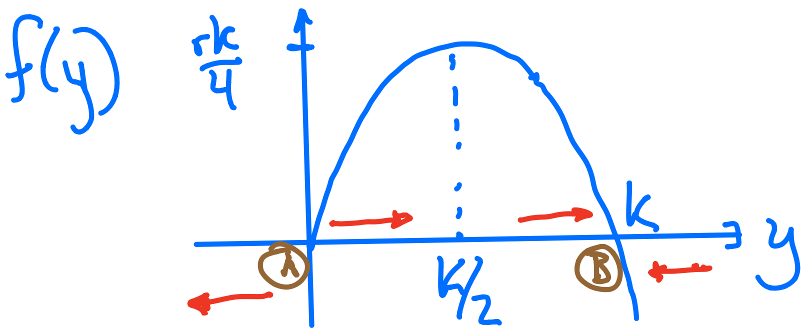
equilibrium pts $\frac{dy}{dt} = 0$

$\therefore \quad y_{eq} = 0 \qquad y_{eq} = \frac{r}{b} \equiv K$

K is "carrying capacity"

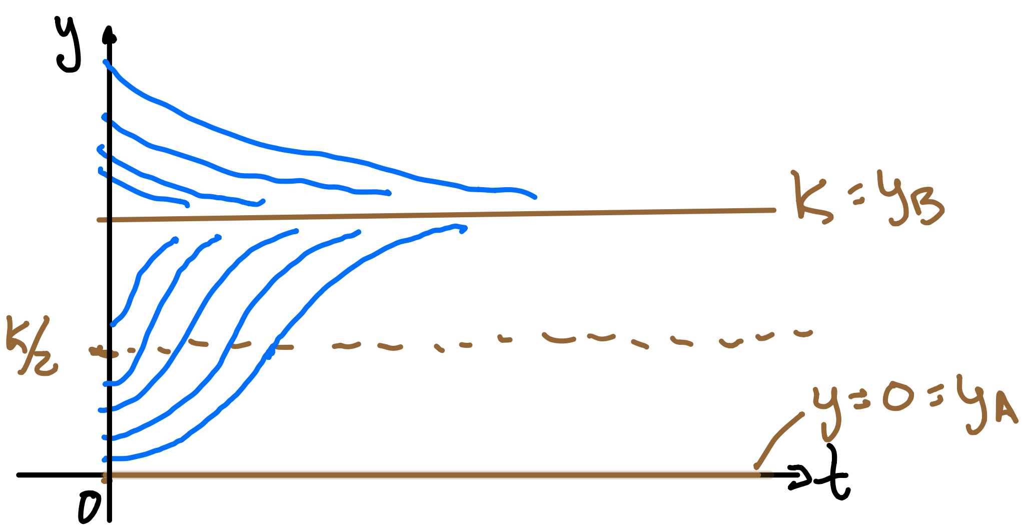
$$\frac{dy}{dt} = ry\left(1 - y/K\right) = f(y)$$

The Logistic Equation




$y_A = 0$ $\quad y_{eq}$ conditionally unstable

$y_B = K$ $\quad y_{eq}$ stable equilibrium



Note: for $y(0)$ small we get a population that grows $\propto ry$, i.e. $f(y) \sim ry$

so the ODE is approximately $\frac{dy}{dt} \approx ry$ for y small.

Note: Look again at $f(y)$, find $\frac{\partial f}{\partial y} = 0$ for $y = K/2$

$$f(y) = ry - ry^2/K$$

$$\frac{\partial f}{\partial y} = r - \frac{2ry}{K} = 0$$

$$\therefore y = K/2$$ the inflection point.

$$f(K/2) = r\frac{K}{2}\left(1 - \frac{K}{2K}\right) = \frac{rK}{4}$$

the value of the largest rate of growth.

# Solution to Logistic Equation

$$\frac{dy}{dt} = ry\left(1 - y/k\right)$$ The Logistic Equation

$$\frac{dy}{y(1 - y/K)} = r\,dt$$ separable

integrate b.s.

$$\underbrace{\int \frac{dy}{y(1-y/k)}}_{} = \int r\,dt = rt + C$$

$$\int \frac{dy}{y(1-y/k)}$$

use partial fractions

$$\frac{1}{y(1-y/k)} = \frac{A}{y} + \frac{B}{1-y/k}$$

$$1 = A(1 - y/k) + By$$

$$\left.\begin{aligned} 1 &= A \\ 0 &= -\frac{A}{k} + B \end{aligned}\right\} \quad A = 1 \quad B = \frac{1}{k}$$

$$\int \frac{dy}{y(1-y/k)} = \int \frac{dy}{y} + \frac{1}{k}\int \frac{dy}{1-y/k}$$

integrating

$$\ln|y| + -\frac{1}{k}\ln\left|(1-y/k)\right|$$

$\ln\left(\frac{y}{(1-y/k)^K}\right) = rt + c$ exponentiating

$\frac{y}{(1-y/k)^k} = De^{rt} \Rightarrow \frac{y}{1-y/k} = DKe^{rt}$

We want to solve for $y$, $D$ will be determined from I.C. (later). $y = (1-y/k)DKe^{rt} = \frac{1}{k}(K-y)DKe^{rt}$

$Ky = (K-y)DKe^{rt}$ $\quad Ky = DK^2e^{rt} - yDKe^{rt}$

$$\therefore \boxed{y = \frac{DK}{D + Ke^{-rt}}} \quad (\ddagger)$$

K is known, r is known. D can be found from I.C.

Assume $y(0)$ is known I.C.

$$y(0) = y_0$$

Using (‡) at $t=0$:

$$y_0 = \frac{DK}{D+K}$$

$$y_0(D+K) = DK$$

$$y_0 D + y_0 K = DK$$

$$D(K-y_0) = y_0 K$$

$$D = \frac{y_0 K}{K-y_0}$$

$$y(t) = \frac{y_0 K}{K-y_0} K \frac{1}{\frac{y_0 K}{K-y_0} + Ke^{-rt}}$$

$$y(t) = \frac{y_0 K}{K - y_0} \, \frac{1}{\frac{y_0}{K - y_0} + e^{-rt}}$$

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0) e^{-rt}}$$

$K \geq 0$ carrying capacity (Given)
$r \geq 0$ reproduction rate (Given)
$y_0 \geq 0$ initial population (Given)

Note that if $y_0 = 0$ then $y(t) = 0$ for all $t$. Also if $y_0 = K$, $y(t) = K$ for all $t$. Finally, for any $y_0 \neq 0$ $\lim_{t \to \infty} y(t) = K$, the carrying capacity.