2.4 Differences between LINEAR & NONLINEAR Equations

Thm: (Existence & Uniqueness) for

$$y' + p(x)\,y = g(x)$$ linear first order

with $p(x)$ & $g(x)$ **continuous** on an open interval $\alpha < x < \beta$. Will have a solution (existence) that is unique. That solution $y(x)$ also satisfies $y(x_0) = y_0$, $\alpha < x_0 < \beta$ (the initial condition).

Pf: Continuity of $p(t)$ and $g(t)$ guarrantee that we find an integrating factor

$I(x)$ & that the solution can be expressed as

$$y = \frac{1}{I(x)} \int^{x} g(s) I(s)\, ds + \frac{1}{I(x)} y_0$$

ex) $(t-1)\frac{dy}{dt} + 3ty = 4$

$y(0) = 1$ here $t_0 = 0$

$$\frac{dy}{dt} + \underbrace{\frac{3t}{(t-1)}}_{p(t)} y = \underbrace{\frac{4}{t-1}}_{g(t)}$$

Here p & g are "singular" at $t=1$.

We will choose $\alpha = -\infty < t < 1 = \beta$ as the domain for which unique solutions exist, since $t_0 = 0$

and $t_0 < \beta$.

Solving: $y' + \frac{3t}{t-1} y = \frac{4}{t-1}$

$$I = e^{\int p(t)\,dt} = e^{3(t + \ln(t-1))}$$

$$= e^{3t} e^{\ln(t-1)^3} = (t-1)^3 e^{3t}$$

$$\frac{d}{dt}(yI) = \frac{I4}{t-1} \quad \text{integrate}$$

$$Iy = 4\int^t \frac{(s-1)^3 e^{3s}}{(s-1)} ds + C$$

$$Iy = 4\int^t (s-1)^2 e^{3s}\, ds + C$$

Integrate by parts twice

$$\left[\, y = \frac{4}{27} \frac{(9t^2 - 24t + 17)}{(t-1)^3} + \frac{ce^{-3t}}{(t-1)^3} \right.$$

for $t < 1$

where $c = -\dfrac{27}{68}$, found by enforcing

$y(0) = 1$.

ex) $y' + \tan t \, y = \sin t$

$y(0) = 2$ } IVP

Determine a range of values of t for which solution exists and is unique.

Here $t_0 = 0$

$p(t) = \tan t$

$g(t) = \sin t$

Here $\sin t$ is continuous for all t

However $p(t)$ is continuous for

$-\frac{\pi}{2} < t < \frac{\pi}{2}$ if t is to include $t_0 = 0$

$\therefore$ IVP has a unique solution in the interval

$-\frac{\pi}{2} < t < \frac{\pi}{2}$

The above theorem applies SPECIFICALLY to first order linear IVP. What can be said of the general first order IVP?

<u>Thm:</u> $\begin{cases} \dfrac{dy}{dx} = f(x,y), \quad y = y(x). \\ y(x_0) = y_0 \end{cases}$

Let f be continuous on a rectangle

$\alpha < x < \beta$ and $\gamma < y < \delta$.

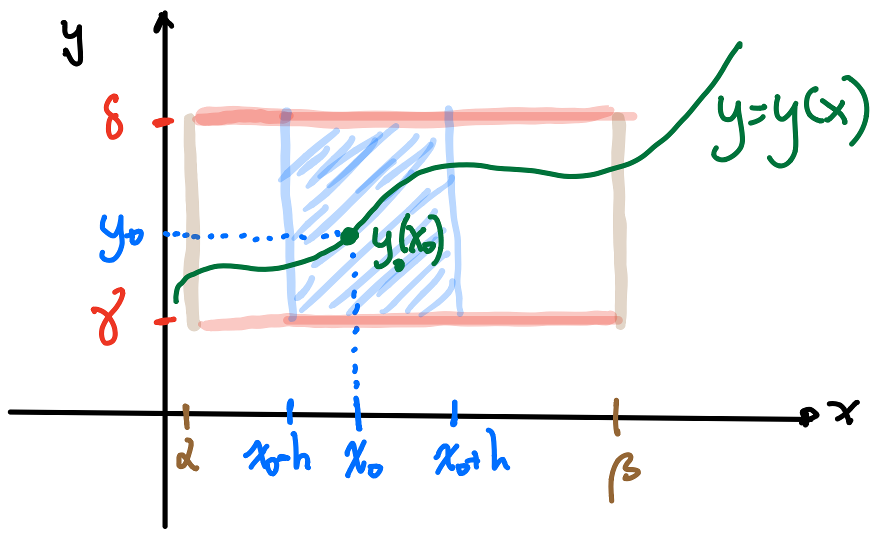
Let $\alpha < x_0 < \beta$.

Then for some interval

$I = (x_0 - h, x_0 + h)$ inside

of (α, β) there exist a solution

$y(x)$ (not necessarily unique):



Th'm: (Uniqueness). Conditions same as above, Further, assume $\frac{\partial f}{\partial y}$ is continuous on $(\alpha,\beta)\times(\gamma,\delta)$ then solution $y(x)$ is unique inside $(\alpha,\beta)\times(\gamma,\delta)$

ex) $\left\{\begin{array}{ll} y' = y^{1/2} & \text{O.D.E.} \\ y(0)=0 & \text{I.C.} \end{array}\right\}$ IVP $(\cancel{\gamma})$

Determine whether the solution exists & is unique over some range (what range?)

Here $f = y^{1/2}$ set $\alpha = -\infty$

$\beta = \infty$

$\gamma = 0$

$\delta = \infty$

The box over which f is defined/continuous.

$x_0 = 0$ is inside $\alpha < x_0 < \beta$.

$\therefore$ Solution exists in "rectangle"

Consider $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}\dfrac{1}{\sqrt{y}}$ is not continuous at $y = 0$.

$\therefore$ solution that passes through 0 is not unique.

In fact, we can solve $(\neq)$

$$y' = y^{1/2} \qquad y(0) = 0$$

$$\frac{dy}{y^{1/2}} = dx \quad \text{separable.}$$

integrate both sides

$$y_1 = \frac{x^2}{2} + C$$

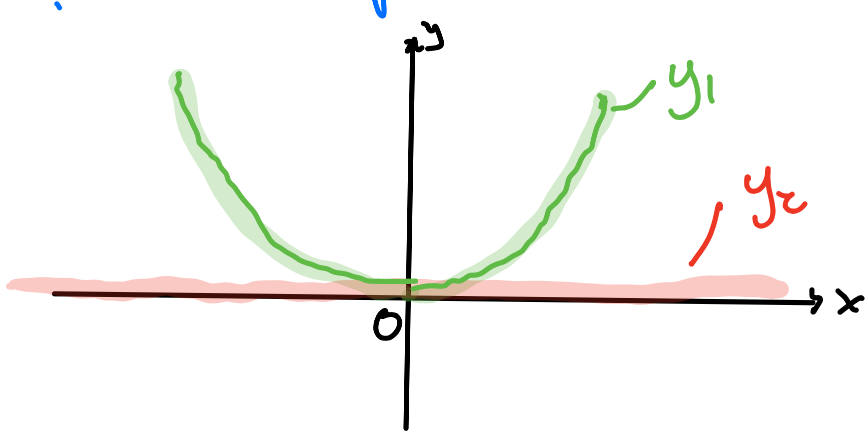
apply I.C. $y(0) = 0$

$$0 = 0 + c \Rightarrow c = 0$$

$$y_1 = \frac{x^2}{2}$$

Also by inspection $y_2 = 0$ is also a solution to $y' = y^{1/2}$ $y(0) = 0$

$\therefore$ not unique solution.

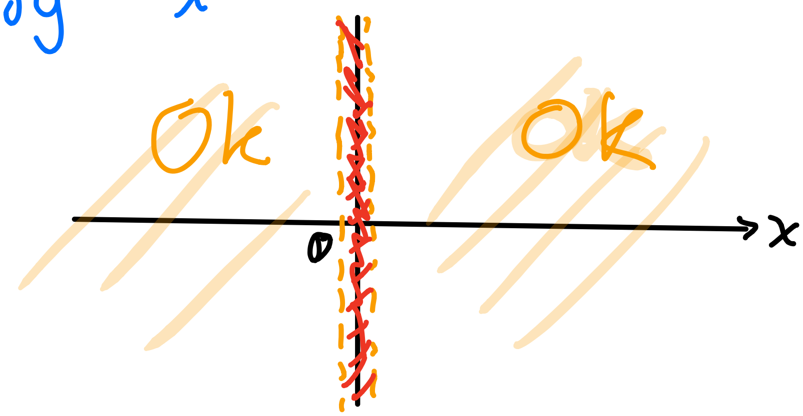


ex) $(\star)\begin{cases} y' = \dfrac{4y}{x} = f(x,y) \\ y(x_0) = y_0 \end{cases}$

Determine whether solution exists & is unique.
For what ranges?

$f(x,y)$ is not defined at $x=0$

$\dfrac{\partial f}{\partial y} = \dfrac{4}{x}$ is not defined at $x=0$



x_0 cannot be 0 due to existence issue.

We can actually solve $(\star)$

$$y' = \frac{4y}{x} \quad \text{separable}$$

$$\frac{dy}{y} = \frac{4}{x}\,dx$$

$$\ln|y| = 4\ln|x| + C$$

or (*) $y = Kx^4$ is the solution

* Note that $x_0 = 0$ leads to $y = 0$, since K is a constant (alternatively $K = 0$)

* Also note that if $x_0 = 0$, we can't find a K non-zero that satisfies (*).

$\therefore \quad y_0 = Kx_0^4 \Rightarrow K = \frac{y_0}{x_0^4}$

so long as $x_0 \neq 0$.

if $x_0 > 0$ set $0 < x < \infty$
$-\infty < y < \infty$

if $x_0 < 0$ set $-\infty < x < 0$
$-\infty < y < \infty$

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Def: Autonomous 1st order ODE. Consider $y = y(x)$, x is independent variable.

then $\frac{dy}{dx} = f(x,y) = f(y)$

(no explicit x dependence in f)

is called an AUTONOMOUS 1st order Diff Eq.

Else $\frac{dy}{dx} = f(x,y)$ is

Non-autonomous.

// ex)

$\frac{dy}{dx} = y^{1/2}$ is autonomous

$\frac{dy}{dx} = \frac{4x}{y}$ is non-autonomous. //

The Stability of Equilibrium Solutions of autonomous ODE's:

$$(\star) \quad \frac{dy}{dt} = f(y) \qquad t \geq 0$$

is an autonomous ODE. As before, we find y_{eq} (equilibrium) solutions by setting $\frac{dy}{dt} = 0$ and then finding the roots of $f(y) = 0$.

Why? because $\frac{dy_{eq}}{dt} = 0$, by definition.

We might find one, many or no real y_{eq}.

We can also examine their stability:

let $y = y_{eq} + \delta$, $(\ddagger)$

the perturbation $\delta(t)$, where $|\delta(t)| << 1$

We say that y_{eq} is stable if $|\delta(t)|$ remains small as $t \to \infty$. Otherwise unstable (or conditionally) stable.

Substitute $(\ddagger)$ into $(\star)$:

$$\frac{d}{dt}\left(y_{eq} + \delta(t)\right) = f\left(y_{eq} + \delta(t)\right)$$

$$0 \quad \frac{dy_{eq}}{dt} = 0$$

$$\frac{d}{dt}\delta(t) = f(y_{eq} + \delta)$$

$$\approx \underbrace{f(y_{eq})}_{0} + \left.\frac{\partial f}{\partial y}\right|_{y_{eq}} \delta$$

$$\therefore \quad \frac{d}{dt}\delta(t) = \left.\frac{\partial f}{\partial y}\right|_{y_{eq}} \delta(t) \equiv \alpha\, \delta(t)$$

$$\alpha = \left.\frac{\partial f}{\partial y}\right|_{y_{eq}} \quad \text{a constant.}$$

$$\frac{d}{dt}\delta(t) = \alpha\,\delta(t)$$

is separable $\frac{d\delta}{\delta} = \alpha\, dt$

$$\ln \delta = \alpha t + c \implies \delta = K e^{\alpha t}$$

So if $Re(\alpha) > 0$ (the real part of α)

$\lim_{t\to\infty} \delta(t)$ is unbounded. UNSTABLE

if $Re(\alpha) < 0$ then $\lim_{t\to\infty} \delta(t) = 0$

STABLE.

So we can assess the stability of an autonomous ODE by looking at the sign of $f(y)$ near y_{eq}:

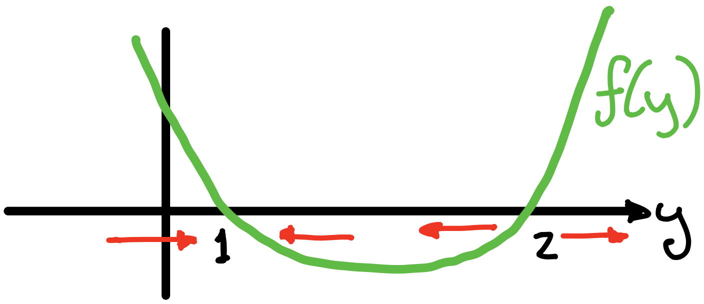
ex) A second order chemical reaction:

$$\frac{dy}{dt} = k(y-1)(y-2) = f(y)$$

Find roots of $f(y)=0$

$y_{eq}=1$ $\quad$ $y_{eq}=2$

PLOT $f(y)$ and infer $\frac{\partial f}{\partial y}$ sign to assess y_{eq} stability:



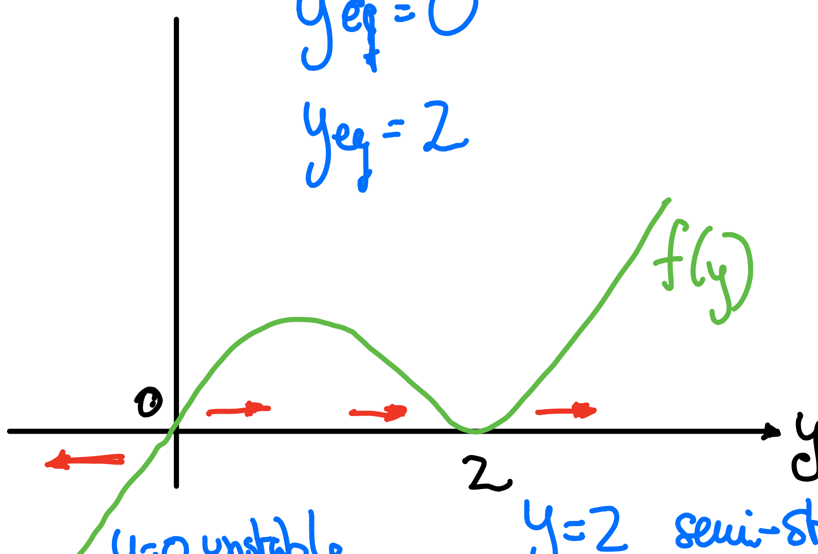
$\therefore\ y_{eq} = 1$ stable

$y_{eq} = 2$ unstable

ex) $\dfrac{dy}{dt} = y(2-y)^2$

$y_{eq} = 0$

$y_{eq} = 2$



$y=0$ unstable $\qquad y=2$ semi-stable