2.3 Modeling with 1st Order ODE's:

Reminder:

(0) Propose basic problem (If not given).

(1) Identify the independent & dependent variables

(2) Units of these.

(3) Identify basic principles involved.

(4) Principles lead to equation(s).

(5) Check units consistency of equation(s).

(6) Solve equation(s).

(7) Check that solution satisfies equation

(8) Check that solution is consistent with basic principles.

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Newton's Law of Cooling

$$\frac{dT}{dt} \propto T - T_0$$

T is temperature
t is time

means "proportional to"

T_0 is a constant. To make this into an equation: look at units

$$\frac{dT}{dt} = \frac{[\text{Temp}]}{[\text{time}]}$$

$$T - T_0 = [\text{Temp}]$$

$$(\$) \qquad \frac{dT}{dt} = k\,[T - T_0]$$

$\therefore$ $k = [1/\text{time}]$ constant.

let's assume $t \geq 0$

The equilibrium solution of ($)

$$\frac{dT_{eq}}{dt} = 0 = k\,(T_{eq} - T_0)$$

there is a $T_{eq} = T_0$.

Solving ($): $$\frac{dT}{T - T_0} = k\,dt$$

Separable. Integrate b.s.

$$\int \frac{dT}{T - T_0} = \int k\,dt$$

$$\ln|T - T_0| = kt + C$$

exponentiate:

($) $$T = K e^{kt} + T_0 \qquad t \geq 0$$

K is determined by I.C.

I.C. $T(t_0) = T_a$ $\qquad t_0 \geq 0$

T_a is given.

k is determined by problem statement

($k \leq 0$ for cooling problems)

ex) A pie is pulled out of oven set at 350°F. After 1 hour, pie temperature is 175°F. The pie can be eaten when temperature is 110°F. The room temperature is 70°F. Q: How long do we have to wait to eat the pie?

let $T(t)$ temp in [°F]

t time in [hrs]

let $T_0 = 70$ (°F) Equilibrium Temp

ODE $\frac{dT}{dt} = k(T-70)$ $\quad [{}^\circ F/hr]$

$k = [1/hr]$

I.C. $T(0) = 350^\circ F$

k must be determinable from the information

"After 1 hr pie is at 175°F"

The solution to ODE is (#)

$$T(t) = Ke^{kt} + 70 \quad t \geq 0$$

Apply I.C. to find K

$$T(0) = 350 = Ke^0 + 70$$

$$K = 350 - 70 = 280$$

(#) $T(t) = 280e^{kt} + 70 \quad t \geq 0$

Find k: After 1hr $T=175$ $\therefore$

Using (☆) $175 = 280e^{k\cdot 1} + 70$

$105 = 280e^{k}$

$\frac{105}{280} = e^{k}$

$k = \ln\left(\frac{105}{280}\right) \approx -0.98$

$T(t) = 280e^{-0.98t} + 70 \quad t \geq 0$

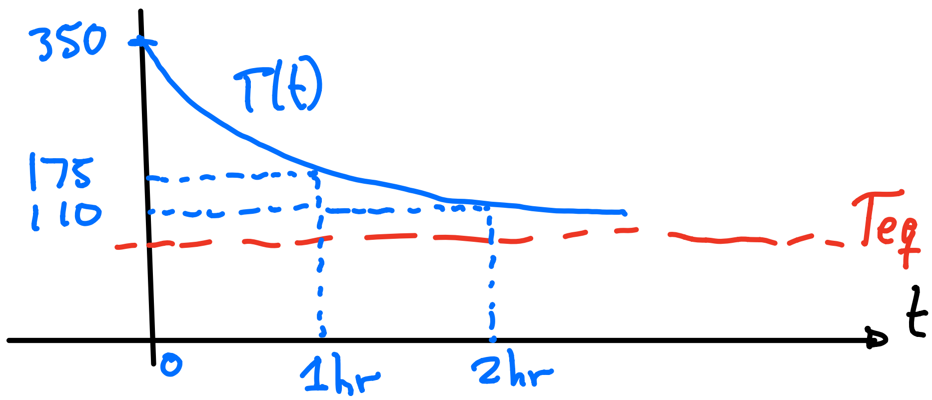
We eat pie at $t = t^*$

$110 = 280e^{-0.98t^*} + 70$

Solving for $t^* \approx 1.98$ hrs

$t^* = \frac{\ln(1/7)}{\ln\left(\frac{105}{280}\right)}$

Check solution makes sense:



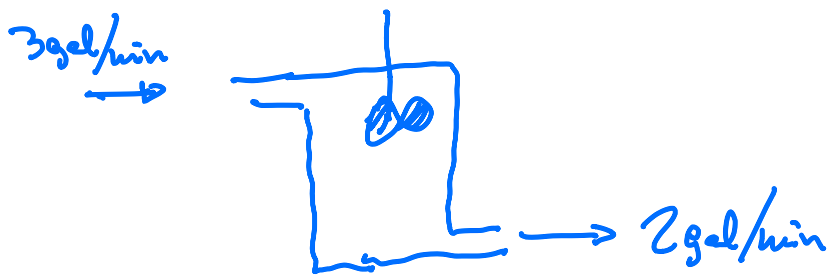
ex) A tank can hold 500 gallons. It has 100 gallons of brine (H_2O & $NaCl$), with 5 lbs of NaCl.

A brine solution containing 1 lb/gal of NaCl is pumped into tank at a rate of 3 gal/min. The well-mixed solution is drained at 2 gal/min.

Find salt in tank after t (≥ 0) minutes:

let $A(t)$ amount of NaCl [lbs]

$t \geq 0$ time [min]



$\left[\frac{\text{lbs}}{\text{min}}\right]$ $\quad \frac{dA}{dt} = \text{Rate in} - \text{Rate out}$

Rate in: $\frac{1\,\text{lb}}{\text{gal}} \frac{3\,\text{gal}}{\text{min}} = 3\,\text{lb/min}$

$$\frac{dA}{dt} = 3 - \text{rate out}$$

Rate out: $\left[\frac{\text{lbs}}{\text{min}}\right]$ of $A(t)$

Accumulation $\propto$ flux difference

$(3-2)$ gal/min $= 1$ gal/min

$$2\,\frac{gal}{min}\;\frac{A}{100+t}\;\frac{lb}{gal} = \frac{2A}{100+t}$$

$$\therefore\; \frac{dA}{dt} = 3 - \frac{2A}{100+t}$$

$$\frac{dA}{dt} + \frac{2A}{100+t} = 3$$

$$\frac{dA}{dt} + \left(\frac{2}{100+t}\right)A = 3$$

1st order linear

$$I = e^{\int \frac{2}{100+t}\,dt} = e^{2\ln(100+t)}$$

$$I = e^{\ln(100+t)^2} = (100+t)^2$$

$$I\frac{dA}{dt} + \left(\frac{2}{100+t}\right)IA = 3(100+t)^2$$

$$\frac{d}{dt}(IA) = 3(100+t)^2$$

integrate b.s.

$$IA = 3\int^{t}(100+s)^2 ds + C$$

$$A(t) = \frac{3}{I}\int^{t}(100+s)^2 ds + \frac{1}{I}C$$

let $w = 100+s$

$dw = ds$

$$\int^{t}(100+s)^2 ds \Rightarrow \int w^2 dw = \frac{1}{3}w^3\Big|^{t} = \frac{1}{3}(100+t)^3$$

$$A(t) = \frac{3}{(100+t)^2}\,\frac{1}{3}(100+t)^3 + \frac{C}{(100+t)^2}$$

$$A(t) = 100+t + c(100+t)^{-2}$$

If $A(0) = 5$ lb, for example

$$5 = A(0) = 100 + \frac{c}{100} \quad \therefore \quad C = -4.95 \cdot 10^4$$

$$A(t) = 100 + t - \frac{4.95 \cdot 10^4}{(t+100)^2}$$

ex) A Pond has 10^6 gal H_2O. Salty water flows in at a rate of $5 \cdot 10^6$ gal/yr and flows out at same rate. Let $C(t)$ be the salt concentration at time t (yrs). Depending on the time of year, the concentration of incoming salt is $2 + \sin 2t$ gram/gal. At $t = 0$ no salt is found in pond. Find $C(t)$:

$$\frac{dC}{dt} = \frac{[\text{grams}]}{[\text{yr}]}$$

$$\frac{dC}{dt} = \text{rate in} - \text{rate out}$$

rate out: $5 \cdot 10^6 \frac{gal}{yr} \frac{c}{10^7}$ grams/gal

$= \frac{c(t)}{2}$ grams/yr

rate in: $5 \cdot 10^6$ gal/yr $(2+\sin 2t) \frac{gram}{gal}$

$$\therefore \frac{dc}{dt} = 5 \cdot 10^6 (2+\sin 2t) - \frac{1}{2} c(t)$$

let $q = \frac{c}{10^6}$ $\therefore c = 10^6 q$

$$\frac{d}{dt}(10^6 q) = 5 \cdot 10^6 (2+\sin 2t) - \frac{10^6}{2} q$$

$$10^6 \frac{dq}{dt} = 10^6 \, 5(2+\sin 2t) - 10^6 \frac{1}{2} q$$

or $\frac{dq}{dt} = 5(2+\sin 2t) - \frac{1}{2} q$

linear 1st order:

$$\frac{dq}{dt} + \frac{1}{2}q = 5(2+\sin 2t) \quad (*)$$

$I = e^{\frac{1}{2}t}$, multiply b.s. of $(*)$:

$$\frac{d}{dt}(qI) = I\,5(2+\sin 2t)$$

integrate:

$$qI = 5\int^t e^{\frac{1}{2}s}(2+\sin 2s)\,ds + K$$

$$q = \frac{5}{I}\left[\int^t 2e^{\frac{1}{2}s}\,ds + \int^t e^{\frac{1}{2}s}\sin 2s\,ds\right] + \frac{K}{I}$$

(3) $$q = 5e^{-\frac{1}{2}t}\left[4e^{t/2} + \int^t e^{\frac{1}{2}s}\sin 2s\,ds\right] + Ke^{-\frac{1}{2}t}$$

$\int^t e^{\frac{1}{2}s}\sin 2s\,ds$ integrate by parts:

$u = \sin 2s \qquad v = 2e^{\frac{1}{2}s}$

$du = 2\cos 2s\, ds \qquad dv = e^{\frac{1}{2}s} ds$

$$\int^t e^{\frac{1}{2}s} \sin 2s\, ds = 2e^{\frac{1}{2}t} \sin 2t - 4 \int^t e^{\frac{1}{2}s} \cos 2s\, ds$$

$u = \cos 2s \qquad v = 2e^{\frac{1}{2}s}$

$du = -2\sin 2s\, ds \qquad dv = e^{\frac{1}{2}s} ds$

$$\int^t e^{\frac{1}{2}s} \cos 2s\, ds = 2e^{\frac{1}{2}t} \cos 2t + 4 \int e^{\frac{1}{2}s} \sin 2s\, ds$$

$\therefore$

$$\int^t e^{\frac{1}{2}s} \sin 2s\, ds = 2e^{\frac{1}{2}t} \sin 2t - 4\left(2e^{\frac{1}{2}t} \cos 2t\right) - 16 \int^t e^{\frac{1}{2}s} \sin 2s\, ds$$

$\therefore$

$$17 \int^t e^{\frac{1}{2}s} \sin 2s\, ds = 2e^{\frac{1}{2}t} \sin 2t - 4\left(2e^{\frac{1}{2}t} \cos 2t\right)$$

or $\int^t e^{\frac{1}{2}s} \sin 2s\, ds = \frac{2}{17} e^{\frac{1}{2}t} \sin 2t - \frac{8 e^{\frac{1}{2}t}}{17} \cos 2t$

Replace into (∋):

$$q = 20 + \frac{10}{17} \sin 2t - \frac{40}{17} \cos 2t + \quad e^{-t/2} K$$

if $q(0) = 0$, we can find K:

$$0 = 20 - \frac{40}{17} - K \Rightarrow \frac{340}{17} - 20 = K$$

$$-\frac{340}{17} = K$$

$$\therefore \quad q = 20 + \frac{10}{17} \sin 2t - \frac{40}{17} \cos 2t - \frac{340}{17} e^{-t/2}$$

Where $q = \frac{c}{10^6}$

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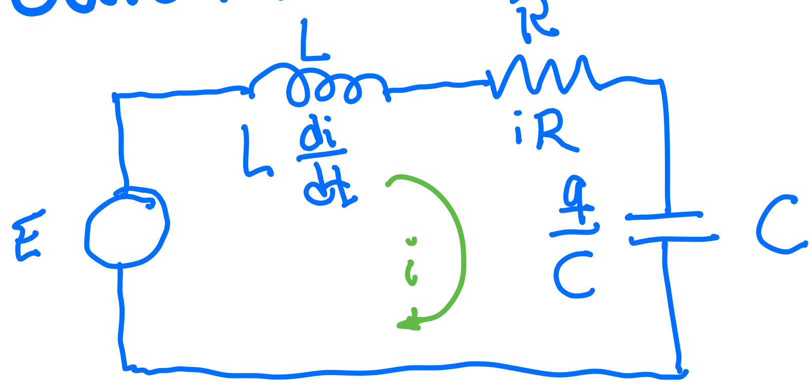
Current in A SERIES CIRCUIT (LRC)



current
$i = \frac{dq}{dt}$
q is charge

DEVICE	**UNITS**
E is EMF	Volts (energy)
L is inductor	Henry
C is capacitor	Faraday
R is resistor	Ω ohms

Principle: Energy Conservation (Ohm's Law Kirchhoff Law)

$$E = L\frac{di}{dt} + Ri + \frac{q}{C} \quad (\#)$$

t time (independent variable)

where $i = \frac{dq}{dt}$

Replacing $i = \frac{dq}{dt}$ then (#)

is

$$\mathcal{E} = L \frac{d}{dt}\left(\frac{dq}{dt}\right) + R\frac{dq}{dt} + \frac{q}{C}$$

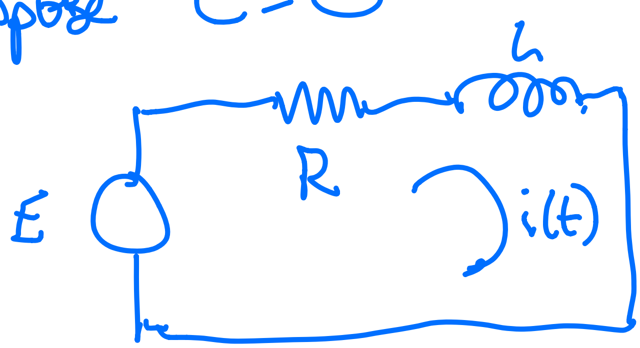
or

$$\mathcal{E} = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} \quad (*)$$

$q = q(t)$ is the dependent variable.

(*) is a 2nd order non-Homogeneous linear ODE. The solution of higher order ODE's will be presented in the next chapter.

Suppose $C = 0$



(LR circuit)

($) $\quad E = iR + L\frac{di}{dt}$ $\quad$ linear first order ODE

$i = i(t)$

ex) $\left\{ \begin{array}{l} E = 100\,V \\ R = 200\,\Omega \\ L = 10\,H \end{array} \right.$ and $i(0) = 1$ Amp I.C.

Using (\$): $\dfrac{E}{L} = \dfrac{R}{L} i + \dfrac{di}{dt}$

or $\dfrac{di}{dt} + \dfrac{R}{L} i = \dfrac{E}{L}$ $(\star)$

let $I = e^{\frac{R}{L}\int dt} = e^{\frac{R}{L}t}$ & multiply b.s.:

$$\frac{d}{dt}(I i) = \frac{E}{L} I$$

integrating

$$I i = \frac{E}{L} \int^{t} e^{\frac{R}{L}s}\, ds + c$$

$$i(t) = \frac{E}{L I} \int^{t} e^{\frac{R}{L}s}\, ds + c/I$$

$$i(t) = \frac{E}{L} \frac{L}{R} e^{\frac{R}{L}t} e^{-\frac{R}{L}t} + c e^{-R/L\, t}$$

$$i(t) = \frac{E}{R} + c\,e^{-R/L\,t}$$

Apply I.C.

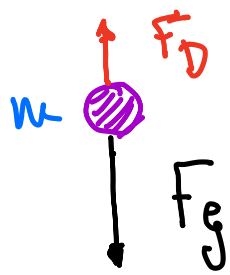
$$i(0) = 1 = \frac{E}{R} + c \Rightarrow c = 1 - \frac{E}{R}$$

$$i(t) = \frac{E}{R} + \left(1 - \frac{E}{R}\right)e^{-R/L\,t} = \frac{1}{2} + \left(1 - \frac{1}{2}\right)e^{-20t}$$

$$\lim_{t\to\infty} i(t) = \frac{E}{R} = \frac{1}{2}$$

FALLING BODIES & DRAG FORCES

Principle: Newton's Second Law



$$ma = F_g - F_D$$

$m = [\text{mass}]$ mass of object

$F_g = [\text{mass} \cdot \text{Length}/\text{Time}^2]$

Gravitational force

F_D = [mass length/time^2]

Drag force

ma = [mass length/time^2] Net downward force

a is the acceleration [length/time^2]

$a = \frac{dv}{dt}$ the rate of change of the velocity

$a = \frac{d^2x}{dt^2}$ the second derivative of the displacement

$F_g = mg$ g is the (constant acceleration due to gravity) = [length/time^2]

$F_D = kv$ since $v = \left[\frac{\text{length}}{\text{time}}\right]$ the units of the $k = \left[\frac{\text{mass}}{\text{time}}\right]$. $k \geq 0$ is a constant

(the assumption is that the velocity is non-negative).

$$\therefore \quad m\frac{dv}{dt} = mg - kv$$

$$(\star) \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

A first order Linear ODE

Compare the form of the equations $(\star)$

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CHEMICAL REACTIONS

"1st order" chemical reaction involves a single chemical:

Let $Q(t)$ be the radioactive content of some chemical. The decay of radioactivity is given by $\frac{dQ}{dt} = kQ$, $k < 0$

A separable equation

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"Second order" chemical reaction

Principle: mass conservation.

Consider a chemical reaction that does not involve heat:

A compound C is formed from compounds A and B. The resulting reaction is such that, for each Kg of A, 4 Kg of B are need to form C. It is observed that 30 Kg of C are formed in 10 minutes. At $t=0$:

$A(0)=50 \quad B(0)=32 \quad C(0)=10$

Determine $C(t)$: the rate is proportional to how much A & B are present:

$$\frac{dC}{dt} \propto \text{amount of } A \,\&\, B$$

To form: $C = A + B$ (mass conservation)

$$(*) \qquad C = A + 4A = 5A$$

so $A = C/5$ and $B = 4C/5$

By $(*)$

$$A(t) = C(t)/5 \quad \text{or} \quad A(t) - \frac{C(t)}{5} = 0$$

at $t=0$ $\quad A(0) - \dfrac{C(0)}{5} = 0$

or $\quad 50 - \dfrac{C(0)}{5} = 0$

at $t=0$ $\quad B(0) - \dfrac{4C(0)}{5} = 0$

$$32 - \frac{4}{5}C(0) = 0$$

$$\therefore \frac{dC}{dt} = k\left(50 - \frac{C(t)}{5}\right)\left(32 - \frac{4C(t)}{5}\right) \quad (\ddagger)$$

the rate of creation of C will be zero if the right hand side is 0.

k is a constant, $[1/\text{min}]$ units.

Factor $\frac{4}{5}$ from the RHS of $(\ddagger)$

$$\frac{dC}{dt} = \frac{4k}{25}(250-C)(40-C)$$

(A)

25.5

$\frac{32.5}{4}$

at $t=0$ $C(0)=0$

Separable 1st order ODE

$$\frac{dC}{(250-C)(40-C)} = \frac{4k}{25}\,dt$$

partial fractions

$$-\tfrac{1}{210}\int \frac{dC}{250-C} + \tfrac{1}{210}\int \frac{dC}{40-C} = \frac{5k}{4}t + \hat{C}$$

$$\frac{1}{210}\ln\left(\frac{40-C}{250-C}\right) = \frac{4kt}{25} + \hat{C}$$

168
840
25
5

$$\ln\left(\frac{250-C}{40-C}\right) = -\frac{210\cdot 4}{25}kt + C =$$

$$= -\frac{168}{5}kt + C$$

exponentiating:

$$(\star)\quad \frac{250-C}{40-C} = De^{-\frac{168}{5}kt} \quad , D \text{ is a constant}$$

$$\therefore\ C(t) = \frac{250-40De^{-\frac{168}{5}kt}}{1-De^{-\frac{168k}{5}t}}$$

We need to determine D and k. We know that $C(0)=0$. Using $(\star)$:

$$\frac{250}{40} = D = \frac{25}{4}$$

$$\frac{250-C}{40-C} = \frac{25}{4}e^{-\frac{168}{5}kt}$$

To find k: $C=30$ when $t=10$ (see problem statement:

$$\frac{250-30}{40-30} = \frac{25}{4}e^{-\frac{168}{5}k10}$$

$$\frac{22\cancel{0}}{1\cancel{0}} = \frac{25}{4}\, e^{-\frac{168}{5}k\cdot 10}$$

$$\frac{88}{25} = e^{-336k}$$

$$\ln\left(\frac{88}{25}\right) = -336k$$

$$\therefore k = -\frac{1}{336}\ln\left(\frac{88}{55}\right) \sim -0.001$$

$$\therefore C(t) = \frac{1000\left(1 - e^{-0.013t}\right)}{25 - 4e^{-0.013t}}$$

Note $\displaystyle\lim_{t\to\infty} C(t) = \frac{1000}{25} = 40$ Kg

Let's also compute the equilibrium solutions to (Δ): set $\frac{dC}{dt} = 0$

then $k(250-c)(40-c) = 0$

So the equilibrium solutions are $C_{eq} = 40,250$. //

COMPOUND INTEREST

Money deposited in an account can make money by accruing interest. There are "simple" and "compound" interest savings accounts and loans.

Compound interest depends on the frequency and on the rate:

let r be the (yearly) rate of interest

let m be the (inverse of frequency) # of times interest is compounded

let t be the time of investment, in years.

let $S(t)$ be the value (in \$) of the account.

An initial deposit of S_0 will be worth

$$S(t) = \left(1 + \frac{r}{m}\right)^{mt} S_0$$

future value

initial value

Why? after year 1:

$$S(1) = \left(1 + \frac{r}{m}\right)^{m} S_0$$

$$S(2) = \left(1 + \frac{r}{m}\right)^{m} S(1) = \left(1 + \frac{r}{m}\right)^{2m} S_0$$

$$S(3) = \left(1 + \frac{r}{m}\right)^{m} S(2) = \left(1 + \frac{r}{m}\right)^{3m} S_0$$

$$\vdots$$

$$(\dagger) \qquad S(t) = \left(1 + \frac{r}{m}\right)^{mt} S_0$$

What is the limit as $m \to \infty$ of

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{m} \equiv L$$

Take the log:

$$\ln L = \ln \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m$$

$$\ln L = \lim_{m\to\infty} m \ln\left(1+\frac{r}{m}\right)^m$$

Let's use L'Hopitals:

$$\ln L = \lim_{m\to\infty}\frac{\ln(1+r/m)}{1/m} = \lim_{m\to\infty}\frac{\frac{1}{1+r/m}\left(-\frac{r}{m^2}\right)}{-1/m^2}$$

$$= \lim_{m\to\infty}\frac{r}{1+r/m} = r$$

Now, exponentiating:

$$e^{\ln L} = e^r \quad \therefore \quad L = e^r$$

What's the point of this calculation?

We want to know what happens as $m\to\infty$ in (‡) because m is the # of times we compound and $m \to \infty$ means that we compound all the time (compound continuously!)

$$\therefore \quad S(t) = \left(1 + \frac{r}{m}\right)^{mt} S_0$$

[$S(t) = S_0 e^{rt}$ if the account is compounded continuously]

ex) Invest S_0 in an account that earns r interest per year. If compounded continuously with a rate of 7.5%, annual, and you want to reach \$1M in 40 yrs. How much do you need to invest? (Note: no money is withdrawn or deposited after the initial deposit)

$$S(40) = 10^6 = S_0 e^{0.075 \cdot 40}$$

$$\therefore S_0 \approx 50{,}000.$$

Determine how much s/year to invest, earning at an annual rate of $r = 7.5\%$, compounded continuously, would be required to reach $S(40) = 10^6$ (a million dollars in 40 yrs).

In this case we are depositing s/year every year:

Year	$
0	s
1	se^r
2	$(s+se^r)e^r$
3	$(s+se^r+se^{2r})e^r$
$\vdots$	$\vdots$
n	$se^r\left[1+e^r+e^{2r}+\cdots e^{(n-1)r}\right]$

So can we estimate

(A) $1+e^r+e^{2r}+\cdots e^{(n-1)r}$

let $x=e^r$ so (A) is

$1+x+x^2+\cdots x^{n-1}$, a geometric series.

$$1+x+x^2\cdots x^{n-1} = \frac{1-x^n}{1-x}$$

$$\therefore \quad se^r\left[1+e^r+e^{2r}\cdots e^{(n-1)r}\right] = \frac{sx(1-x^n)}{1-x}$$

So if $r=0.075$ and $n=40$ then $x=e^{0.075}\approx 1.08$

$$\therefore \quad s(1.08)\frac{(1-1.08^{40})}{1-1.08} = 10^6$$

hence $s \approx \$3786$ needs to be deposited per year.