Section 2.2 Seperable ODE (1st order)

General Form of a Seperable 1st order ODE:

$$G(y)\,dy = H(x)\,dx$$

(£)

so $y = y(x)$

Start with

$$\frac{dy}{dx} = f(x,y)$$

should be expressible as

$$\frac{dy}{dx} = g(y)\,h(x)$$

$$\Rightarrow \frac{dy}{g(y)} = h(x)\,dx$$

has the form ($\mathcal{Y}$)

$G(y) = \frac{1}{g(y)}$ & $H(x) = h(x)$

METHOD OF SOLUTION: INTEGRATE BOTH SIDES

ex) $y' = -\frac{x}{y}$ ODE

$y(0) = 5$ I.C.

Find the particular solution IVP (initial value problem)

① See if ODE is separable 1st order:

$\frac{dy}{dx} = -\frac{x}{y}$ or

$y\,dy = -x\,dx$

so of the form ($\mathcal{Y}$)

$\therefore$ separable

② Integrate b.s:

$$\int y\,dy = -\int x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C \qquad (\star)$$

Now apply I.C. $y(0)=5$

$$\frac{1}{2}5^2 = -\frac{1}{2}0^2 + C$$

$$\therefore C = \frac{25}{2} \qquad (\text{replacing into } \star)$$

$$\therefore \frac{1}{2}y^2 + \frac{1}{2}x^2 = \frac{25}{2}$$

or $y^2 + x^2 = 5^2$ a circle!

An Ap lication: A container with a capacity of 100 gallons holds a brine solution: the liquid consists of water H_2O and salt. The salt concentration of brine entering the tank is 0.05 lb/gal. The rate of inflow of the brine is 3 gal/min. The well-stirred mixture leaves the tank at a rate of

3 gal/min.

(1) Find ODE describing the total salt

(2) If, at $t=0$ (time), tank has 0.1 lbs of salt /gal, how much salt is in the tank after 60 minutes?

(3) What is the asymptotic salt content?

* The tank has 100 gal capacity
* Let $A(t)$ be the amount of salt at time t
* units $A(t) = [lb]$, $t = [min]$
* I.C. $A(0) = 0.1 \frac{lb}{gal} \cdot 100 gal = 10 lb$

$\therefore A(0) = 10$

* The "mass conservation principle" states that the rate of change of A inside the tank should be equal to the rate of inflow of A minus the rate of outflow of A:

$$\boxed{\frac{dA}{dt} = \text{rate in} - \text{rate out}} \qquad \left[\frac{lb}{min}\right]$$

Find rate in:

$\left[\frac{lb}{min}\right]$ $\quad \frac{3 gal}{min} \times 0.05 \frac{lbs}{gal}$

$= 0.15\ lbs/min$

$\left[\frac{lb}{min}\right]$ rate out: $\frac{3 gal}{min} \times \frac{A}{100} \frac{lbs}{gal}$

$= \frac{3}{100} A\ lbs/min$

$\therefore$ IVP $\begin{cases} \frac{dA}{dt} = 0.15 - \frac{3}{100} A & \text{ODE} \\ A(0) = 10 & \text{I.C.} \end{cases}$

Here $A = A(t)$ is the salt in lb in the tank.

This IVP is separable. To see this:

Multiply ODE by dt:

$$dA = (0.15 - 0.03A)\, dt$$

Divide by $0.15 - 0.03A$

to get

$$\frac{dA}{0.15-0.03A} = dt$$ separable

factoring -0.03 from denominator of LHS:

$$\frac{dA}{A-5} = -0.03\,dt$$

Integrate b.s.:

$$\ln|A-5| = -0.03t + C$$

exponentiate b.s:

$$e^{\ln|A-5|} = e^{-0.03t}\underbrace{e^{C}}_{K \text{ (a constant)}} = Ke^{-0.03t}$$

$$A-5 = Ke^{-0.03t}$$

$$\therefore\ A(t) = 5 + Ke^{-0.03t}$$

To find K use initial conditions (I.C.)

$$A(0) = 10 = 5 + Ke^{-0.03\cdot 0}$$

$$5 = Ke^{0} = K$$

$$\therefore\ A(t) = 5\left(1+e^{-0.03t}\right)$$

the solution.

To find salt at $t=60$:

$$s(60) = 5\left(1+e^{-0.03\cdot 60}\right) \approx 5.8316$$

What happens to s as $t\to\infty$?

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} 5\left(1+e^{-0.03t}\right) = 5$$

ex)

$$\frac{dy}{dt} = t-1+ty-y$$

$$\frac{dy}{dt} = t(1+y)-(1+y) = (t-1)(1+y)$$

$$\frac{dy}{1+y} = (t-1)\,dt$$

$$\ln|1+y| = \frac{1}{2}t^2 - t + C$$

exponentiating:

$$1+y = e^{\frac{1}{2}t^2-t}e^{C} = Ke^{\frac{1}{2}t^2-t}$$

$$y = Ke^{\frac{1}{2}t^2-t}-1$$

ex) IVP $\begin{cases} y' = y^2 & \text{ODE} \\ y(0) = 1 & \text{I.C.} \end{cases}$

$$y = y(t)$$

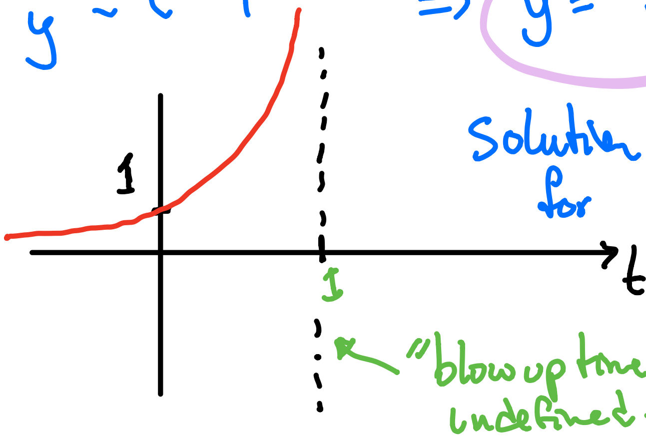
$$\frac{dy}{y^2} = dt \quad \text{or} \quad y^{-2}dy = dt$$ separable

$$-\frac{1}{y} = t + c$$

using the I.C. $y(0)=1$:

$$-\frac{1}{1} = 0 + c \quad \Rightarrow \quad c = -1$$

$$-\frac{1}{y} = t - 1 \quad \Rightarrow \quad y = \frac{1}{1-t}$$



Solution is defined for $t < 1$

"blow up time": solution becomes undefined.

HOMOGENEOUS 1st ORDER ODE:

Definition: A function $f(x)$ is said to be homogeneous of degree n if, by replacing every x in $f(x)$ by λx, where λ is a constant, we find that

$$f(\lambda x) = \lambda^n f(x)$$

ex) $f(x,y) = (2x^2 - 3y^2 + 4xy)$

is homogeneous since $x \to \lambda x$ & $y \to \lambda y$

$$f(\lambda x, \lambda y) = (2\lambda^2 x^2 - 3\lambda^2 y^2 + 4\lambda x \lambda y)$$

$$= \lambda^2 (2x^2 - 3y^2 + 4xy) = \lambda^2 f(x, y))$$

see that $f(\lambda x, \lambda y) = \lambda^2 f(x, y)$ "degree 2"

Homogeneous, 1st order, Differential Equation

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is "homogeneous 1st order ODE" if

$$M(\lambda x, \lambda y) = \lambda^n M(x, y)$$

$$N(\lambda x, \lambda y) = \lambda^n N(x, y)$$

Method of Solution: let $y = ux$

solve for $u(x)$ then go back to $y(x)$

Rule: if the 1st order ODE

$$\frac{dy}{dx} = f(x,y)$$

can be written as

$$\frac{dy}{dx} = g\left(y/x\right), \text{ let } u = y/x$$

the substitution will lead to a separable eq in $u(x)$ //

ex) $\underbrace{(x^2+y^2)}_{M(x,y)}dx + \underbrace{(x^2-xy)}_{N(x,y)}dy = 0$

(1) Check to see whether M & N are homogeneous & of same degree:

$$M(x,y) = x^2+y^2$$

$$M(\lambda x,\lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2 M(x,y)$$

$$N(\lambda x,\lambda y) = \lambda^2 x^2 - \lambda x \lambda y = \lambda^2 (x^2 - xy)$$

so yes!

② Since Homogeneous 1st order, can use trick:

let $y = u(x)\,x$

find $\frac{dy}{dx} = \frac{du}{dx}x + u$

$\therefore$ $\boxed{dy = x\,du + u\,dx}$ (*)

Replace $y = ux$ & dy as in (*) into

$$(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$$

$$(x^2 + u^2x^2)\,dx + (x^2 - xux)[x\,du + u\,dx] = 0$$

Manipulate to get separable eq for u, x:

$$\cancel{x^2}(1 + u^2)\,dx + \cancel{x^2}(1 - u)[x\,du + u\,dx] = 0$$

$$(1 + u^2)\,dx + (1 - u)x\,du + (1 - u)\,u\,dx = 0$$

$$\left[1+u^2+(1-u)u\right]dx+(1-u)x\,du=0$$

$$\left[1+u^2+u-u^2\right]dx+(1-u)x\,du=0$$

$$(1+u)\,dx+(1-u)x\,du=0$$

$$\frac{1-u}{1+u}\,du=-\frac{dx}{x}$$

separable
integrate b.s.

L.H.S divide $1-u/1+u$

$$\left[-1+\frac{2}{1+u}\right]du=-\frac{dx}{x}$$

$$-u+2\ln|1+u|=-\ln|x|+C$$

$$2\ln|1+u|+\ln|x|=u+C$$

$$\ln|1+u|^2+\ln|x|=u+C$$

$$\ln\left|(1+u)^2 x\right| = u + c$$

subs $u = y/x$

$$\ln\left|\left(1+ y/x\right)^2 x\right| = \frac{y}{x} + c$$

$$\left(1+\frac{y}{x}\right)^2 x = Ke^{y/x}$$

$$\left(\frac{x+y}{x}\right)^2 x = \frac{(x+y)^2}{x} = Ke^{y/x}$$

$$(x+y)^2 = Kxe^{y/x}$$

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ANOTHER TRICK FOR 1st ORDER ODE's:

Suppose the ODE has the form

$$\frac{dy}{dx} = f(ax+by+c)$$

a, b, c are constants. $b \neq 0$

trick let $u = ax + by + c$

ex) $\frac{dy}{dx} = \frac{1-x-y}{x+y}$

let $u = 1-x-y$ (*)

$x+y = 1-u$ (***)

$\frac{dy}{dx} = \frac{u}{1-u}$ (**)

but $du = -dx - dy$ ($)

differentiating (*)

Subst ($), into (**)

$dy = du + dx$

$\frac{du + dx}{dx} = \frac{u}{1-u}$

$$\frac{du}{dx} + 1 = \frac{u}{1-u}$$

$$\frac{du}{dx} = \frac{u}{1-u} - 1 = \frac{u-1+u}{1-u} = \frac{2u-1}{1-u}$$

$$\frac{du(1-u)}{2u-1} = dx$$

dividing

$$-\frac{1}{2}du + \frac{1}{4}\frac{du}{(u-\frac{1}{2})} = dx$$

$$-\frac{1}{2}u + \frac{1}{4}\ln\left|u-\frac{1}{2}\right| = x + C$$

$$-\frac{1}{2}(1-x-y) + \frac{1}{4}\ln\left|\frac{1}{2}-x-y\right| = x + C$$

$$1 - \frac{1}{2}x - y = -\frac{1}{4}\ln\left|\frac{1}{2}-x-y\right| + C$$