Section 2.1 Linear 1st ORDER ODE

1ode
Linear

$$\underline{y' + p(t)y} = g(t)$$

$$y = y(t)$$

METHOD OF SOLUTION:

Key observation: let $I(t)$
"integrating factor"

$$\frac{d}{dt}(Iy) = I'y + Iy'$$

We are going to find an $I(t)$ such that

$$I y' + p(t) y I = \frac{d}{dt}(I y)$$

Using **1ODE**:

(★) $$\underbrace{I y' + p(t) y I} = g(t) I$$

$$\frac{d}{dt}(I y) = g(t) I$$

Take $I = e^{\int p\,dt}$

Note $\frac{dI}{dt} = p I$

$$\frac{d}{dt}\left(e^{\int p\,dt} y\right) = \underbrace{e^{\int p\,dt}}_{I} y' + \underbrace{p e^{\int p\,dt}}_{I'} y$$

So (★)

$$\frac{d}{dt}\left(e^{\int p\,dt}\,y\right) = g(t)I(t)$$

Integrate both sides:

$$e^{\int p\,dt}\,y = \int_0^t g(s)\,I(s)\,ds + C$$

$$\therefore\ y(t) = e^{-\int p\,dt}\int_0^t g(s)\,I(s)\,ds + e^{-\int p\,dt}\,C$$

THE SOLUTION

ex) $y' + 2ty = t$ (€)

here $p(t) = 2t$ i.e. $y' + p\,y = g$

$g(t) = t$

So this is 1st order linear

Integrating Factor $I(t) = e^{\int p\,dt} = e^{2\int t\,dt} = e^{t^2}$

Multiply B.S. by I

$$\underbrace{y' e^{t^2} + 2te^{t^2} y}_{} = te^{t^2}$$

$$\frac{d}{dt}\left(e^{t^2} y\right) = te^{t^2}$$

Integrate b.s.

$$e^{t^2} y = \int_0^t s e^{s^2} ds + c$$

$$y = e^{-t^2} \int_0^t s e^{s^2} ds + ce^{-t^2}$$

let $u = s^2$

$du = 2s\,ds$

$$y = e^{-t^2} \int_0^u e^u \frac{du}{2} = e^{-t^2} \frac{1}{2} e^{t^2} + ce^{-t^2}$$

$$y = \frac{1}{2} + ce^{-t^2} \quad (\neq)$$

To check: plus $(\neq)$ into $(\in)$

to see if it satisfies ODE.

ex) $ty' - 4y = t^6 e^t$

Δ $\quad y' - \frac{4}{t} y = t^5 e^t$

is $y' + p(t)y = g(t)$? YES 1st order ODE

$p(t) = -\frac{4}{t} \qquad g(t) = t^5 e^t$

$$I = e^{\int p\,dt} = e^{-4\int \frac{dt}{t}} = e^{-4\ln t}$$

$$= e^{\ln t^{-4}} = t^{-4}$$

Multiply (A) by I

$$y' t^{-4} - \underbrace{\frac{4}{t} t^{-4} y}_{} = t^{-4} t^{5} e^{t}$$

$$\frac{d}{dt}(yI) = te^{t}$$

Integrate b.s.

$$yI = \int_0^t se^{s}\,ds + C$$

$$y = I^{-1}\int_0^t se^{s}\,ds \quad I^{-1}C \qquad (\#)$$

Calculation: $\displaystyle\int_0^t se^{s}\,ds = \underbrace{se^{s}}_{uv}\Big|^{t} - \underbrace{\int^{t} e^{s}\,ds}_{\int v\,du}$

$u = s \qquad v = e^{s}$

$du = ds \qquad dv = e^{s}ds$

$$\int se^{s}ds = te^{t} - \int^{t} e^{s}ds = (t-1)e^{t}$$

Substitute back into (#)

Also $I^{-1} = (t^{-4})^{-1} = t^4$

$\therefore$

$$y = I^{-1}(t-1)e^t + I^{-1}c$$

$$y = t^4(t-1)e^t + t^4 c$$

$$y = t^4\left[c + (t-1)e^t\right]$$

Let's check

$$\frac{dy}{dt} = 4t^3\left[c + (t-1)e^t\right] + t^4 \frac{d}{dt}\left[(t-1)e^t\right]$$

$$= 4t^3\left[c + (t-1)e^t\right] + t^4\left[e^t + (t-1)e^t\right]$$

Substitute into A:

$$\frac{dy}{dt} = \frac{4}{t}y + t^5 e^t$$

$$4t^3\left[c+(t-1)e^t\right]+t^4\left[e^t+(t-1)e^t\right]$$

$$\overset{?}{=}\frac{4}{t}t^4\left[c+(t-1)e^t\right]+t^5e^t$$

$$=4t^3\left[c+(t-1)e^t\right]+t^5e^t$$

so

$$t^4e^t+(t-1)e^tt^4\overset{?}{=}t^5e^t$$

$$t^4e^t+t^5e^t-t^4e^t\overset{?}{=}t^5e^t \quad \text{Yes}$$

so solution checks out. //

Another example:

ex) $\cos t\, y'+\sin t\, y=1$

divide by $\cos t$

(*) $y' + \tan t\, y = \sec t$

$I = e^{\int \tan t\, dt} = e^{-\ln(\cos t)}$

$= e^{\ln(1/\cos t)} = \frac{1}{\cos t} = \sec t$

Multiply (*) by I :

$\sec t\, y' + \sec t \tan t\, y = \sec^2 t$

$\therefore \frac{d}{dt}\left(y \sec t\right) = \sec^2 t$

integrating b.s. :

$y \sec t = \int^t \sec^2(s)\, ds + C$

$y = \cos t \int^t \sec^2(s)\, ds + \cos t\, c$

$y = \cos t \tan t + \cos t\, c$

$\therefore\ y = \sin t + \cos t \cdot c$