SECTION 1.2

SOME SOLUTIONS TO ODES

What constitutes a solution to an ODE? IT SATISFIES THE DIFFERENTIAL EQUATION

Other questions:

Also, does every ODE have a solution?
NO.

Does the solution have to be unique?
NO.

Take $\frac{dx}{dt} = kx.$ (*)

It has a solution of the form

(†) $x(t) = x_0 e^{kt}$. To see this:

$$\frac{dx}{dt} = k x_0 e^{kt} \stackrel{?}{=} kx = k\left(x_0 e^{kt}\right).$$

this proves that (†) is the solution

to ($\ni$).

Is either $A\sin(\omega t)$ or $B\cos(\omega t)$ a solution to

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad ?$$

[ω is a constant.
A, B constants.

Take $x = A\sin\omega t$

$x' = A\omega\cos\omega t$

$x'' = -A\omega^2\sin\omega t$

Substitute into $x'' + \omega^2 x = 0$

$$-A\omega^2\sin\omega t + \omega^2 A\sin\omega t = 0$$

Check on your own to see if $B\cos\omega t$ is a solution

ODE'S & SOME OF THEIR SOLUTIONS MAY APPLY TO DIFFERENT PROBLEMS IN NATURE.

The following problems have the same mathematical structure and solutions with different interpretations

Take (*) $\frac{dv}{dt} = g - \beta v$ (from Newton's 2nd Law)

wind drag ($\beta \geq 0$)

gravity

acceleration of body

(**) $\frac{dp}{dt} = rp - k$ (reproducing mice in an environment with finite food supply)

growth rate

carrying capacity

Here $p(t)$ is the mice population [#]

t is time

r is a rate [1/time]

k is [#/time]

These 2 equations are the "same" and have the "same" solution. However, The solutions have different interpretations.

Take $\frac{dp}{dt} = rp - k \quad (\star)$

and ① Find equilibrium solutions if they exist

② Solve $(\star)$

③ Examine the stability of the equilibrium solutions.

① Set $\frac{dp_e}{dt} = 0$ & attempt to find p_e.

$\boxed{p_e = \frac{k}{r}}$ Equilibrium solution

Require $r \neq 0$, since

$p_e \geq 0$, require $k \geq 0$ & $r > 0$ OR $k \leq 0$ & $r < 0$

② Solution of $(\star)$:

$$\frac{dp}{dt} = rp - k$$

$$\frac{dp}{rp-k} = dt, \text{ or } \frac{dp}{p - k/r} = r\,dt$$

Integrate both sides:

$$r\int dt = rt + c_1 \quad \text{and}$$

$$\int \frac{dp}{p-k/r} = \ln\left|p - \frac{k}{r}\right| + c_2$$

$$\ln\left|p - \frac{k}{r}\right| = rt + c$$

exponentiate $e^{rt+c} = e^{c}e^{rt}$

$$p - \frac{k}{r} = Ce^{rt}$$

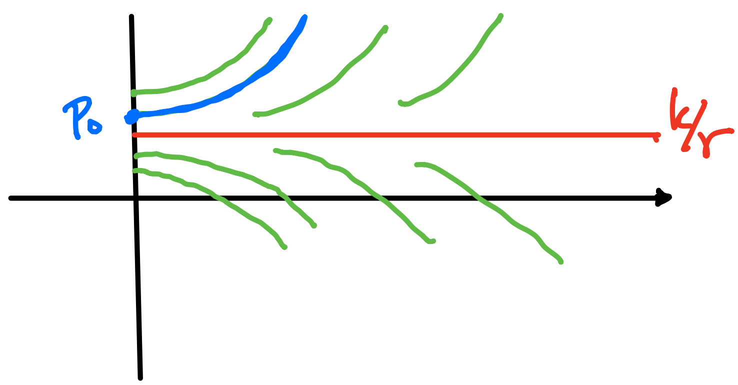
$$\boxed{p = Ce^{rt} + \frac{k}{r}} \quad t \geq 0$$

Examine solution for $r>0$ $k>0$

Assume $P(0) = P_0$, the initial population.

$$P(0) = P_0 = Ce^0 + \frac{k}{r} \Rightarrow C = P_0 - \frac{k}{r}$$

$$\therefore \quad P(t) = \left(P_0 - \frac{k}{r}\right)e^{rt} + \frac{k}{r}$$



3 Stability: we examine the stability of $P_e = k/r$ $(r>0)$

This case is asymptotically unstable because the limit as $t \to \infty$ of a solution starting at

$P_e \pm \delta$ moves away from P_e:

$\lim_{t \to \infty} p(t)$ is undefined.

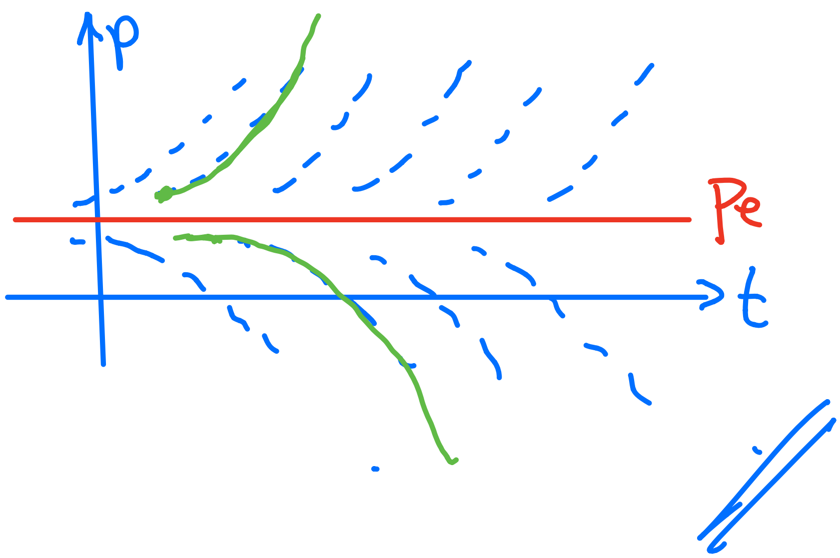
CAN SEE THIS FROM SLOPE FIELD:

$f(p) = rp - k$

$r > 0$

$k > 0$

Slope field



Examine Solution $r < 0$ $k < 0$

$$p(t) = \left(p_0 - \frac{k}{r}\right)e^{rt} + \frac{k}{r}$$

The $\lim_{t \to \infty} p(t) = \frac{k}{r}$ asymptotically stable

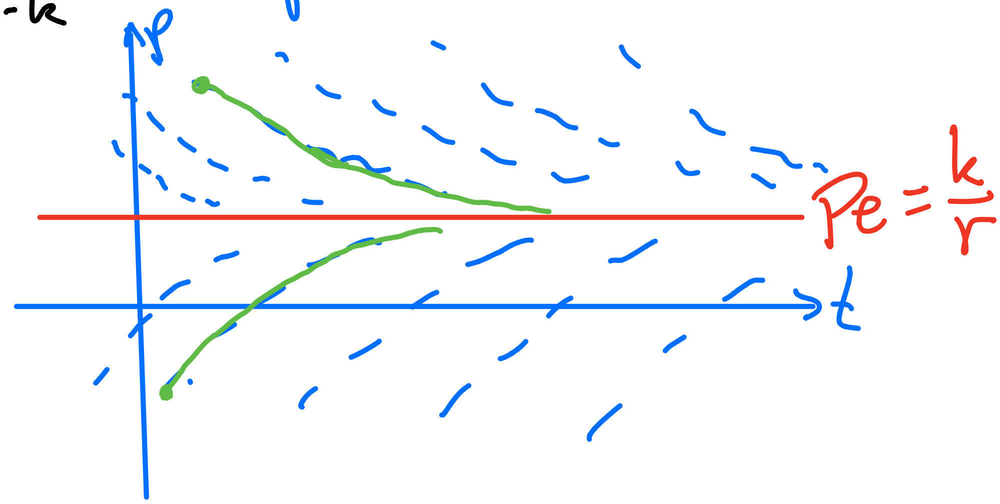
regardless of p_0

The slope field $r<0$ $k<0$

$f(p) = rp - k$

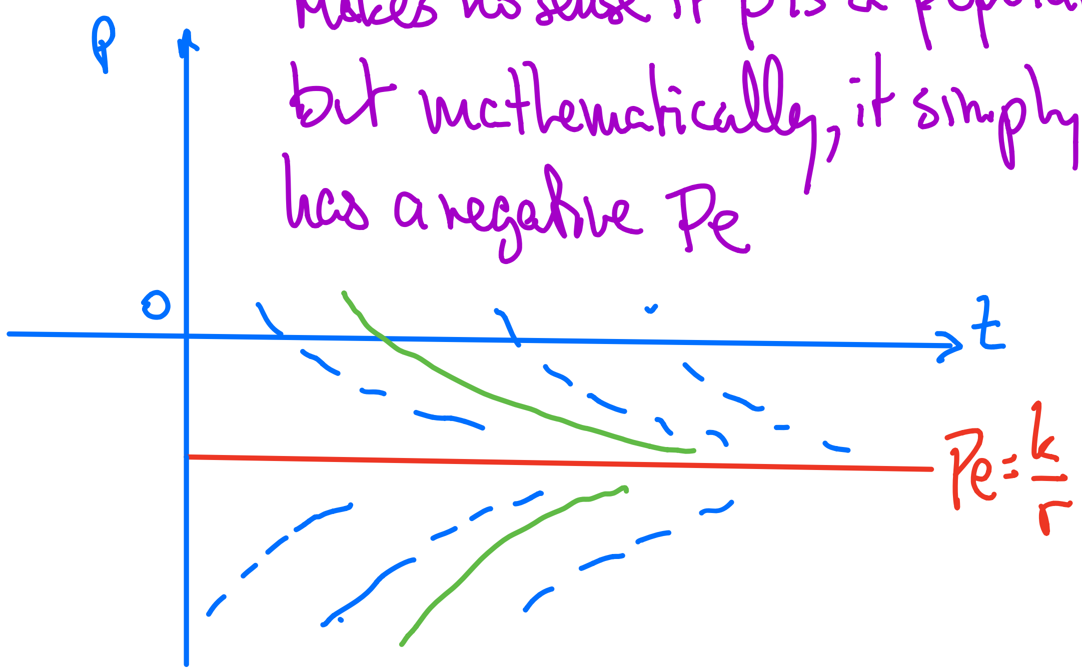
$r<0$

$k>0$



The slope field for $r<0$ $k<0$

Makes no sense if p is a population but mathematically, it simply has a negative P_e



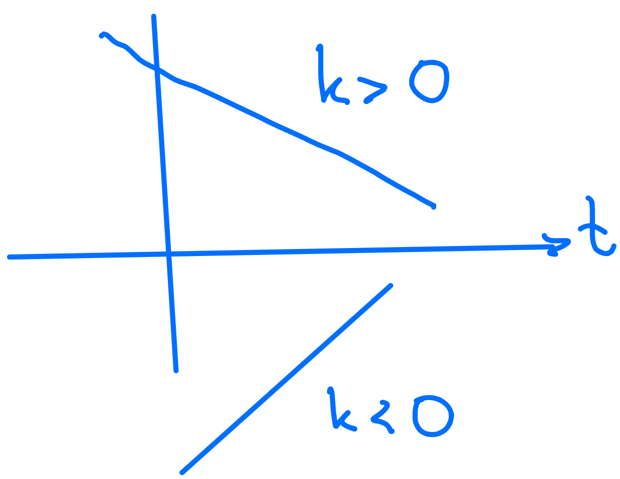
asymptotically stable.

What about $r=0$?

$P_e = \frac{k}{r}$ makes no mathematical sense

$$\frac{dp}{dt} = -k$$

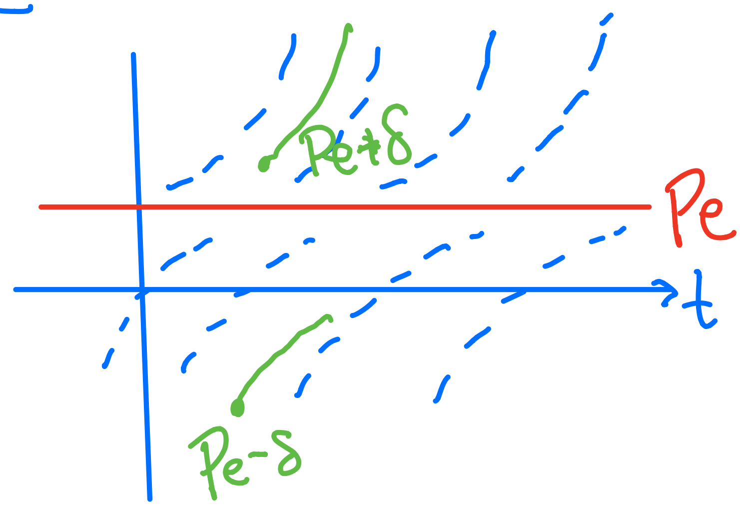
$$p(t) = -kt + C$$



No equilibrium solution.

Another type of stability is called conditionally stable (or conditionally unstable)

The phase portrait would be

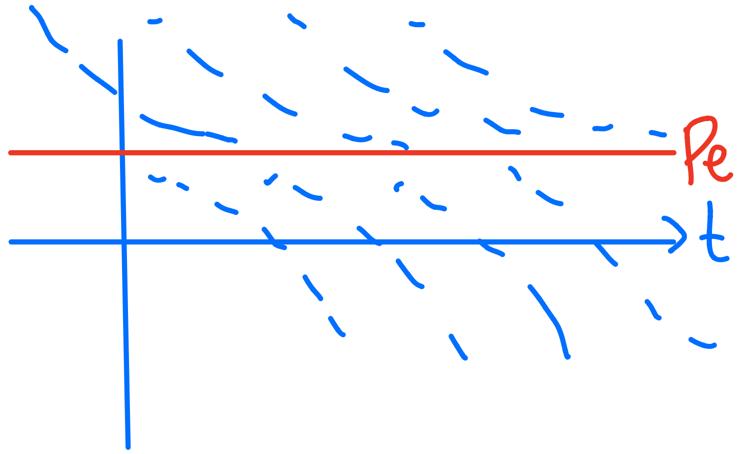


Here Pe is stable for Pe-δ

Pe is unstable for Pe+δ

$\delta > 0$

or another scenario is



stable for Pe+δ

unstable for Pe-δ

δ>0