MATH 255 MIDTERM 2

NAME:

Instructions: You must include all the steps in your derivations/answers. Reduce answers as much as possible, but use *exact arithmetic*. Write neatly.

---

1. (25 %) Consider the force $\mathbf{F}(x,y) = -y\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$. Calculate the work done on a particle along the triangular path described below. The work done by this force on an particle is found by calculating $\oint \mathbf{F} \cdot d\mathbf{r}$. The path is clockwise in the x,y-plane and consists of straight lines. The starting point of the path is $(0,0)$ then it is at a vertex $(1,1)$, then $(1,0)$ and finally back to the origin.

   As a line integral $-\int y dx + \int x^2 dy = -7/6$, since:

   a: $(0,0) \to (1,1)$:
   $$\int_0^1 -x dx + \int_0^1 y^2 dy = -1/2 + 1/3$$

   b: $(1,1) \to (1,0)$:
   $$\int_1^0 1 dy = -1$$

   c: $(1,0) \to (0,0)$:
   $$\int_1^0 0 dx + \int_0^0 x^2 dy = 0$$

2. (25 %) For $f(x, y, z) = 1 + 4xyz$, the point $P(-1, 1, 1)$, the vector $\hat{\mathbf{u}} = -\frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

- Compute ∇f.
- Evaluate ∇f at P.
- Find the unit vector that gives the direction of steepest increase at P.
- Find the directional derivate in the direction $\hat{\mathbf{u}}$ at P.

Answer:
$\mathbf{p} := \nabla f = 4yz\hat{\mathbf{i}} + 4xz\hat{\mathbf{j}} + 4xy\hat{\mathbf{k}}$. Evaluated at P yields $\mathbf{p}(P) = 4(\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$. The steepest ascent will be in the direction $\mathbf{p}$. At P, the unit normal pointing in the direction of steepest increase is $\hat{\mathbf{p}}(P) = \mathbf{p}/|\mathbf{p}|(P) = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}})$. The directional derivative in the direction $\hat{\mathbf{u}}$ at P is the dot product $\hat{\mathbf{u}} \cdot \mathbf{p}(P) = \frac{4}{\sqrt{3}}$.

3. (25 %) Compute the following line integral:

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds,$$

with $\mathbf{F}(x, y, z) = x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$. The path C described by the displacement vector $\mathbf{r} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, for $0 \leq t \leq 1$. **Hint:** use the chain rule on the integrand.

Answer:
$\int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds = \int_0^1 \mathbf{F} \cdot \mathbf{v} dt$, $\mathbf{v} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$, $\mathbf{F} \cdot \mathbf{v} = t - 2t^3$. The integral $\int_0^1 \mathbf{F} \cdot \mathbf{v} dt = \left(\frac{1}{2}t^2 - \frac{1}{2}t^4 \right|_0^1 = 0$

4. (25 %) For $-\infty < t < \infty$, consider the displacement vector $\mathbf{r} = \hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$.

   (a) Draw a picture of the path traced out by the displacement vector. Add to the figure a few representative realizations of the displacement vector for $t > 0$, $t < 0$, and $t = 0$.

   (b) Find the unit tangent vector $\mathbf{T}$

   (c) Find the unit normal vector $\mathbf{N}$.

Hint: Consider separately $t > 0$, $t < 0$, and $t = 0$.

$\mathbf{T} = \mathbf{r}'/|\mathbf{r}'| = \hat{\mathbf{j}}$. Also, $v = \frac{\partial s}{\partial t} = 2|t|$. Hence $\mathbf{T} = \text{sgn}(t)\hat{\mathbf{j}}$. Note that it is undefined at $t = 0$.

The vector $\mathbf{N} = \frac{\partial \mathbf{T}}{\partial s}/|\frac{\partial \mathbf{T}}{\partial s}|$. But $\frac{\partial \mathbf{T}}{\partial s} = \frac{\partial \mathbf{T}/\partial t}{\partial \mathbf{t}/\partial s} = \frac{1}{v}\frac{\partial \mathbf{T}}{\partial t}$. The derivative of $\text{sgn}(t)$ is $2\delta(t)$. So $\mathbf{N} = \mathbf{0}$ for $t \neq 0$. At $t = 0$ it is undefined. We say either that $\mathbf{N}$ is undefined for all time, or we say it is defined for all time except for at $t = 0$