MTH255 EXAM 1

NAME:

Instructions: You must include all the steps in your derivations/answers. Reduce answers as much as possible, but use *exact arithmetic*. Write neatly.

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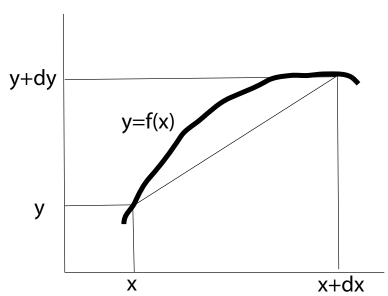
1. (30%) Derive the arc length formula

$$s = \int_a^b [1 + (f')^2]^{1/2}\, dx,$$

where the segment is described by the function $y = f(x)$ and $f'(x) = df/dx$. The segment spans from $x = a$ to $x = b$. **Hint**: use the following figure:



Answer: **SEE CLASS NOTES**.

Take a piece of the segment, from x to $x + \Delta x$. Then the piece has approximate length $\Delta s = [\Delta x^2 + \Delta y^2]^{1/2} = [1 + \Delta y^2/\Delta x^2]^{1/2}\Delta x$. Then take the limit as $\Delta x \to 0$ and $\Delta y \to 0$, to get $ds = [1 + (dy/dx)^2]^{1/2} dx$. Finally,

$$s = \int_{x=a}^{x=b} ds = \int_a^b [1 + f'^2]^{1/2} dx.$$

for the length of the curve described by $y = f(x)$, from $x = a$ to $x = b$.

2. (20%) Consider the orbit, traced out by $\mathbf{r}(t) = \langle -\sin(t), \cos(t), 1\rangle$, with $t \in (-\infty, \infty)$. Find the values of t when $\mathbf{r}$ and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ are orthogonal.

Answer:

$\mathbf{v}(t) = \langle -\cos(t), -\sin(t), 0\rangle$. Then $\mathbf{r} \cdot \mathbf{v} = 0$, is

$$\cos(t)\sin(t) - \cos(t)\sin(t) = 0,$$

which is true for any t. Hence the vectors are orthogonal for all time.

3. (20%) Find an equation to the tangent plane to the surface $\sin(xyz) = \frac{1}{2}$, at the specific point $P = (\pi, 1, 1/6)$.

   Answer: The gradient of the level set $\phi := \sin(xyz) - \frac{1}{2} = 0$ is

$$< yz\cos(xyz), xz\cos(xyz), xy\cos(xyz) > .$$

   At the point P the gradient evaluates to $< 1/6\, b, \pi/6\, b, \pi\, b >$ where $b := \cos(\pi/6) = \frac{\sqrt{3}}{2}$. The tangent plane is thus

$$\frac{b}{6}(x - \pi) + \frac{b\pi}{6}(y - 1) + \pi b(z - \frac{1}{6}) = 0.$$

4. (30%) A ball's acceleration is $\mathbf{a}(t) = \langle 0, -g(1-e^{-t}) \rangle \quad \text{m/s}^2$. where $t \geq 0$ is time in seconds. Find the

   (a) (10%) velocity $\mathbf{v}(t)$, $t \geq 0$.

   (b) (10%) The displacement $\mathbf{r}(t)$, $t \geq 0$.

   (c) (10%) The *maximum height* in the object's flight.

   - Let $g = 10$ m/s^2.
   - The initial conditions of displacement and velocity, respectively, are $\mathbf{r}_0 = \langle 0, 0 \rangle$ m, and $\mathbf{v}_0 = \langle 1, 20 \rangle$ m/s.

   Answer: this is an accelerating reference frame, the "gravitational" force is time dependent: only as t gets large does the acceleration approach a constant g.

   Integrating the acceleration and using $\mathbf{v}_0$:

$$\mathbf{v}(t) = < 1, 30 > + < 0, -g(t + e^{-t}) > = < 1, 30 - g(t + e^{-t}) >$$

   integrating again and applying initial conditions $\mathbf{r}_0$,

$$\mathbf{r}(t) = < 0, -g > + < t, 30t > + < 0, -g(\frac{1}{2}t^2 - e^{-t}) > = < t, -g + 30t - g(\frac{1}{2}t^2 - e^{-t}) >$$

   The maximum height is achieved when the vertical component of the velocity is zero. Hence, when

$$3 - t = e^{-t},$$

   which happens when $t \approx 3$. We replace this value of t into the y-component of the displacement to get

$$y_{max} \approx 35.5\, m.$$

   The actual value is a bit more than 35.51 m.