MATH 255 FINAL

Instructions: You must include all the steps in your derivations/answers. Reduce answers as much as possible, but use *exact arithmetic*. Write neatly, please: I will dock 10 points for careless presentation of any kind. *Scientists and engineers uphold very high standards of ethics: the work you submit in this exam must be yours. Be prepared to explain your answers in person. Also, please document your take home final: keep all of your calculations (work not included in your final exam submission until you get a final grade in your class).*

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1. (25 pts) The gravitational field $\mathbf{F}$ of a planet of mass m, placed at the origin, is given by

$$\mathbf{F} = -Gm\frac{\mathbf{r}}{r^3},$$

   where r is the length of $\mathbf{r}$, G is the gravitational constant. Show that the flux of the gravitational field through the sphere of radius a is independent of a. **Hint**: consider a region bounded by 2 concentric spheres.

2. (15 pts) Assume g is a scalar function and $\mathbf{f}$ a vector function. Use the identity $\nabla \cdot [g\mathbf{f}] = \nabla \cdot \mathbf{f}g + \nabla g \cdot \mathbf{f}$ to compute the divergence of $\mathbf{F} = -\frac{\mathbf{r}}{r^3}$. Here $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and $r = \sqrt{x^2+y^2+z^2}$, the magnitude of $\mathbf{r}$.

3. (10 pts) Decide whether the work done by the force, in 3-d space, $\mathbf{F} = (x+y)\mathbf{i}$ is path independent (all that matters is the starting and ending points.

4. (15 pts) Find the circulation $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the perimeter of the triangle given by the plane $2x+y+z=2$, in the first octant. The field $\mathbf{F} = 3xz\hat{\mathbf{j}}$.

5. (10 pts) Find the *total* flux of $\mathbf{F} = x^2y^2z\mathbf{k}$ where S is a closed cone, described by $\sqrt{x^2+y^2} = z$ with $0 \le z \le R$.

6. (25 pts) The speed of a tornado or whirlpool is a decreasing function of the distance from its center. Assume a velocity $\mathbf{v} = K(-y\hat{\mathbf{i}}+x\hat{\mathbf{j}})/r^2$ where $r^2 = x^2+y^2$, and K is a constant.

   - Sketch the vector field with $K = 1$ and separately, with $K = -1$.
   - Determine the speed, the magnitude of the velocity, as a function of the distance to the center of the vortex.
   - Compute the divergence of $\mathbf{v}$ and its curl.
   - Compute the circulation of $\mathbf{v}$ counterclockwise about the circle of radius R at the origin.
   - The computations above show that the curl is zero but the circulation is not. Explain why this does not contradict Stokes' Theorem.
   - Find conditions on $\mathbf{F}$ that associate it to the different components of the exact differential.

7. (30 pts) Let $\rho(x,y,z) = z^3$ be the density of a solid. Compute its total mass $\int_V \rho dV$, where V is bounded by $z=0$, $z = 4+\sin(2x)+\cos(2y)$, and $-\pi \le x \le \pi$ and $-\pi \le y \le \pi$. **Hint:** exploit the Divergence Theorem.

8. (30 pts) Consider a rectangular region D of the $x-y$ plane that excludes the origin. Find p such that the circulation on the perimeter of the region D is zero, for

$$\mathbf{F} = \frac{y^3}{r^p}\hat{\mathbf{i}} - \frac{xy^2}{r^p}\hat{\mathbf{j}},$$

where $r^2 = x^2 + y^2$.

9. (40 pts) Let R be a region in a plane that has a unit normal $\hat{\mathbf{n}} = \langle a, b, c\rangle$ and boundary C. Let $\mathbf{F} = \langle bz, cx, ay\rangle$.

   (a) Show that $\nabla \times \mathbf{F} = \hat{\mathbf{n}}$.

   (b) Show that the area of R is given by

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

   (c) Consider a curve C given by $\mathbf{r} = \langle 5\sin t, 13\cos t, 12\sin t\rangle$, for $0 \le t \le 2\pi$. Prove that C lies in a plane by showing that $\mathbf{r} \times \frac{d\mathbf{r}}{dt}$ is constant for all t.

   (d) Use part (b) to find the area of the region enclosed by C in part (c). Hint: find the unit normal consistent with the orientation of C.