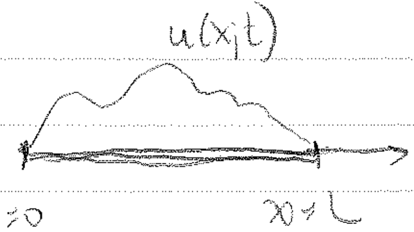
1D Heat Equation

(I) Finite domain



$$T = T_0 + u(x,t)$$

ν is a constant

P.D.E. $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2}$ $\qquad 0 \leq x < L$, $t \geq 0$

I.C. $u(x,0) = g(x)$ $\qquad 0 < x < L$

B.C. Consider (A) $\Big\{\, u(0,t) = u(L,t) = 0$

Dirichlet

(B) $\Big\{\, \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0$

Neumann

For both cases, try separation of variables

$$u(x,t) = \varphi(t)\,\psi(x)$$

substitute in D.E.

$$\frac{\varphi_t}{\varphi} = \nu \frac{\psi_{xx}}{\psi},$$

after dividing by $\varphi\psi$.

$$\frac{\varphi_t}{\nu\varphi} = \frac{\psi_{xx}}{\psi} = -k^2$$

$$\begin{cases} \varphi_t = -k^2\nu\varphi \rightarrow \varphi = \varphi_0 e^{-k^2\nu t} \\ \psi_{xx} + k^2\psi = 0 \rightarrow \psi = A\cos kx + B\sin kx \end{cases}$$

Case (A) let $u = u_A = \varphi\psi_A$

$$\psi_A = A\cos kx + B\sin kx$$

$$\psi_A(0) = \psi_A(L) = 0 \rightarrow \psi_A(x) = B_n \sin\frac{n\pi x}{L}$$

$$k_n = \frac{n\pi}{L} \quad n = 1, 2 \ldots$$

$$\therefore \quad u_A = \sum_{n=1}^{\infty} B_n e^{-k^2\nu t} \sin\frac{n\pi x}{L}$$

$$u_A(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

multiply both sides by $\sin\frac{m\pi x}{L}$ and integrate wrt x from 0 to L:

$$\int_0^L \sin\frac{m\pi x}{L} g(x)\, dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx$$

But
$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \frac{L}{2}\delta_{nm}$$

where $\delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n\neq 0 \end{cases}$

$$\therefore \int_0^L \sin\frac{n\pi x}{L}\, g(x)\,dx = B_n \frac{L}{2} \qquad n=1,2\ldots$$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

For B: let $u = u_B = \varphi\,\psi_B$

$$\psi_B = A\cos kx + B\sin kx$$

$$\psi_B' = -kA\sin kx + Bk\cos kx$$

$$\psi_B'(0) = Bk \Rightarrow B = 0$$

$$\psi_B'(L) = A\cos kL = 0 \Rightarrow k_n = \frac{(n+\frac{1}{2})\pi}{L} \quad n=0,1,2.$$

$$u_B = \sum_{n=0}^{\infty} B_n' \cos\left[\frac{(n+\frac{1}{2})\pi}{L}x\right] e^{-\nu\left[\frac{(n+\frac{1}{2})\pi}{2}\right]^2 t}$$

$$u_B(x,0) = g(x) = \sum_{n=0}^{\infty} B_n \cos k_n x$$

$$k_n = (n+\tfrac{1}{2})\pi/L$$

Multiply both sides by $\cos\left[\frac{(m+\frac{1}{2})\pi}{L} x\right]$ & integrate dx from 0 to L:

$$\int_0^L g(x)\cos\frac{(m+\frac{1}{2})\pi x}{L}\,dx = \sum_{n=0}^{\infty} B_n \int_0^L \cos\frac{(n+\frac{1}{2})\pi}{L}x \cos\frac{(m+\frac{1}{2})\pi}{L}x\,dx$$

$$\begin{cases} B_n = \dfrac{2}{L}\displaystyle\int_0^L g(x)\cos\frac{(n+\frac{1}{2})\pi x}{L}\,dx & n=1,2\ldots \\ B_0 = \dfrac{1}{L}\displaystyle\int_0^L g(x)\,dx \end{cases}$$

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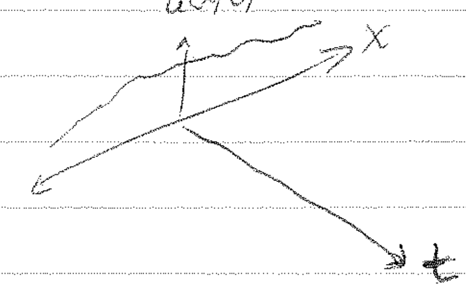
II The Whole Real Line



$$T = T_0 + u(x,t)$$

P.D.E. $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2}$ $\quad x \in \mathbb{R}$, $t > 0$

I.C. $u(x,0) = g(x)$

Assume $\displaystyle\int_{-\infty}^{\infty} |g(x)|\, dx < \infty$ "absolutely integrable"

$$\begin{cases} u(x,t) \to 0 \text{ as } x \to \pm\infty \\ \dfrac{\partial u}{\partial x}(x,t) \to 0 \text{ as } x \to \pm\infty \end{cases}$$

Conditions permit use of Fourier transform.
We'll also make use of Convolutions.

$$\text{(i)} \begin{cases} u(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{u}(k,t)\, e^{-ikx}\, dk \\[2ex] \hat{u}(k,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx \end{cases}$$

Note: unlike before $\hat{u}$ depends on both k & t.

Substitute (1) into PDE

$$\frac{\partial \hat{u}}{\partial t}(k,t) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t) e^{ikx}\,dx = \int_{-\infty}^{\infty} \nu \frac{\partial^2 u}{\partial x^2}(x,t) e^{ikx}\,dx$$

(3) $$\therefore \quad \frac{\partial \hat{u}(k,t)}{\partial t} = -\nu k^2 \int_{-\infty}^{\infty} u(x,t) e^{ikx}\,dx = -\nu k^2 \hat{u}(k,t)$$

since $\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) = -k^2 \mathcal{F}(u)$

Take F.T. of I.C.

(4) $$\mathcal{F}(g(x)) = \hat{g}(k)$$

So in Fourier space, solve the ODE problem (3) & (4):

$$\begin{cases} \frac{\partial \hat{u}}{\partial t}(k,t) = -\nu k^2 \hat{u}(k,t) \\ \hat{u}(k,0) = \hat{g}(k) \end{cases}$$

$$\hat{u}(k,t) = e^{-\nu k^2 t} g(k)$$

Take $\mathcal{F}^{-1}(\hat{u}(k,t))$:

$$(5) \quad u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\hat{g}(k) e^{-\nu k^2 t}}_{\hat{u}(k,t)} e^{-ikx} \, dk$$

this is the final answer.

But we can make the solution more explicit by using convolution.

let $\hat{w}(k,t) = e^{-k^2 \nu t}$, then in (5)

$$\hat{u}(k,t) = \hat{g}(k) e^{-\nu k^2 t} = \hat{g}(k)\, \hat{w}(k,t)$$

a product of 2 FT is a convolution in real space; so

$$(5a) \quad u(x,t) = \int_{-\infty}^{\infty} g(p)\, w(x-p,t)\, dp$$

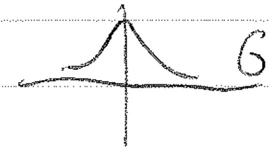
Formally, $w(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{w}(k,t) e^{-ikx} \, dk$

$$\text{or} \quad w(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 \nu t} e^{-ikx} \, dk \quad (6)$$

Let's try to do this integral. Rewrite

$$e^{ikx} = \cos(kx) - i\sin(kx) \quad \text{so}$$

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\nu k^2 t}\cos kx\,dk - \underbrace{\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\nu k^2 t}\sin kx\,dk}_{0}$$

Since $e^{-\nu k^2 t}$  Gaussian bump (even)

then $e^{-\nu k^2 t}\sin kx$ is odd.

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\nu k^2 t}\cos kx\,dk = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\nu k^2 t}\cos kx\,dk$$

let $k = \frac{z}{\sqrt{\nu t}}$ $\therefore$ $dk = \frac{dz}{\sqrt{\nu t}}$

(8) $\therefore$ $$w(x,t) = \frac{2}{\sqrt{\nu\pi t}}\int_{0}^{\infty} e^{-z^2}\cos\left(\frac{xz}{\sqrt{\nu t}}\right)dz = \frac{1}{\sqrt{4\pi\nu t}}\, e^{-x^2/4\nu t}$$

(Table of integrals)

subst (8) into (5a)

$$u(x,t) = \int_{-\infty}^{\infty} f(p)\, w(x-p,t)\,dp = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} f(p)\, e^{-(x-p)^2/4\nu t}\,dp$$