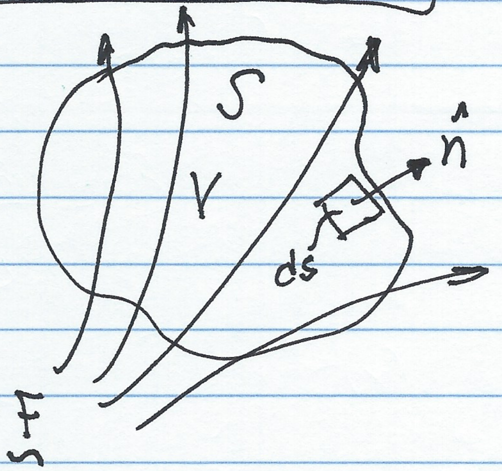
14.8 THE DIVERGENCE THEOREM
(GAUSS' THEOREM)

$$\boxed{\int_V \nabla\cdot\underset{\sim}{F}\, dV = \oint_S \underset{\sim}{F}\cdot d\underset{\sim}{S}}$$

S is the closed surface, enclosing V

$\hat{n}$ is the outward normal

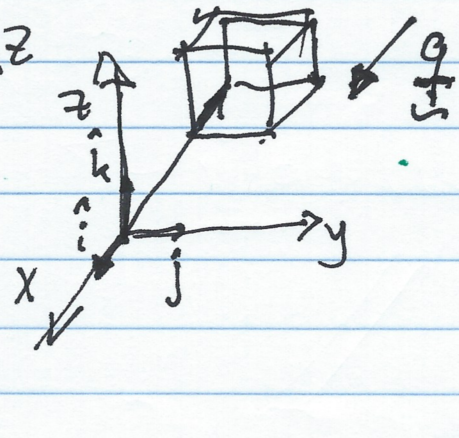


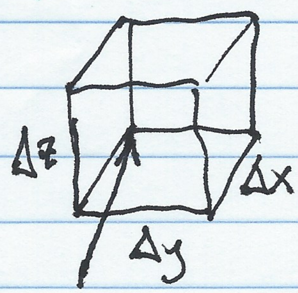
To get some intuition, look at simplest possible problem:

let $\underset{\sim}{q} = \hat{i} f(x)$

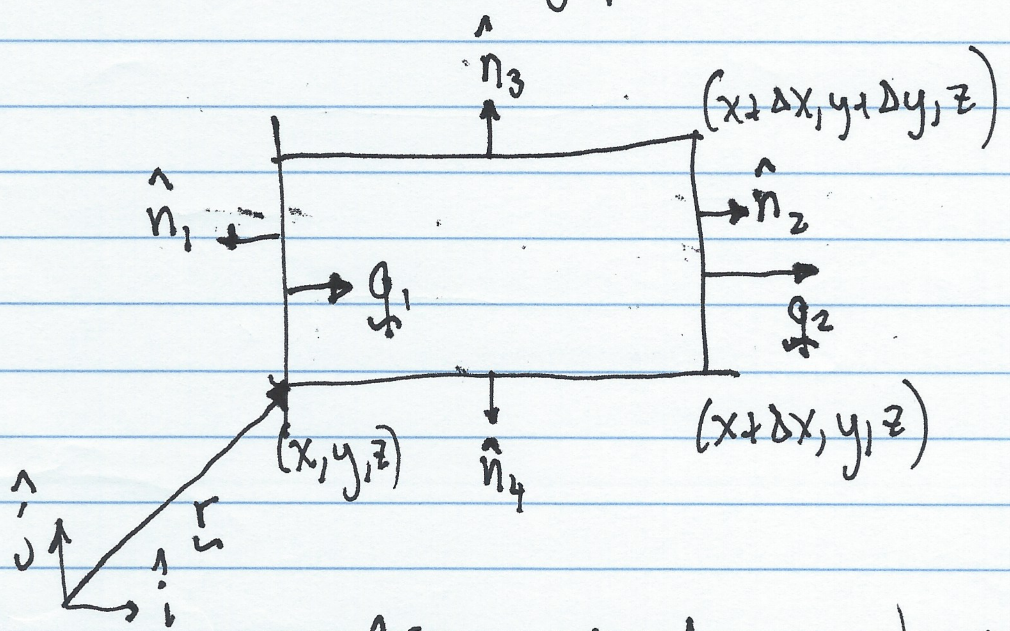
V is going to be a box located at (x,y,z)

and size $\Delta x \times \Delta y \times \Delta z$





Side View in xy plane



$$\begin{cases} \mathbf{q}_1 = \hat{i} f(x) & \hat{n}_1 = -\hat{i} \\ \mathbf{q}_2 = \hat{i} f(x+\Delta x) & \hat{n}_2 = \hat{i} \end{cases} \quad \Bigg| \quad \begin{matrix} \hat{n}_3 = \hat{j} & \mathbf{q}_3 = \hat{i} f(x) \\ \hat{n}_4 = -\hat{j} & \mathbf{q}_4 = \hat{i} f(x) \end{matrix}$$

Let's compute ~~differences~~ total of fluxes entering & exiting the box:

$$\mathbf{q}_2 \cdot \hat{n}_2 \, dy\, dz + \mathbf{q}_1 \cdot \hat{n}_1 \, dy\, dz$$

(no fluxes entering/leaving in the y direction because $\mathbf{q} = f(x)\hat{i}$, i.e. aligned in the x direction

$$\left[f(x+\Delta x)\, \hat{i}\cdot\hat{i} - f(x)\, \hat{i}\cdot\hat{i} \right] dy\, dz$$

$$= \left[f(x+\Delta x) - f(x) \right] dy\, dz$$

but $f(x+\Delta x) \approx f(x) + \Delta x \dfrac{df}{dx} + \dfrac{1}{2}\Delta x^2 \dfrac{d^2 f}{dx^2} + \dots$

$\therefore \left[f(x+\Delta x) - f(x) \right] dy\,dz$

$= \left[f(x) + \Delta x f' + \frac{1}{2}\Delta x^2 f'' + \dots - f(x) \right] dy\,dz$

if $|\Delta x| << 1$ then

$\approx \left[f(x) + \Delta x \dfrac{df}{dx} - f(x) \right] dy\,dz$

$= \dfrac{df}{dx} \Delta x\, dy\, dz \equiv \dfrac{df}{dx} dx\,dy\,dz = \dfrac{df}{dx} dV$

More generally, suppose $\underset{\sim}{q} = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$

The total fluxes entering/leaving box is what's entering & leaving sides 1 & 2, which is,

$$\frac{\partial f(x,y)}{\partial x} dV$$

plus how much enters/leaves sides 3 and 4:

$$\left(\underset{\sim}{q}_3 \cdot \hat{n}_3 + \underset{\sim}{q}_4 \cdot \hat{n}_4 \right) dx\,dz = \left[g(x, y+\Delta y) - g(x,y) \right] dx\,dz$$

$$\approx \frac{\partial g}{\partial y} dy\,dx\,dz = \frac{\partial g}{\partial y} dV$$

Consider the most general case:

$$\underset{\sim}{q} = f(x,y,z)\,\hat{i} + g(x,y,z)\,\hat{j} + h(x,y,z)\,\hat{k}$$

Total fluxes will be $\left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right] dV$

$$= \underset{\sim}{q}\cdot\hat{n}\,dS$$

Integrating over a volume enclosed by S:

$$\int_V (\nabla\cdot\underset{\sim}{q})\,dV = \oint_S \hat{q}\cdot d\underset{\sim}{S} \qquad d\underset{\sim}{S} = \hat{n}\,dS$$

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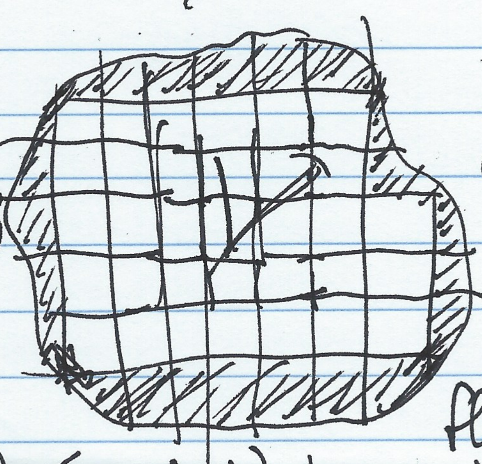
Another way to make sense of this:

S 

Take a closed volume and grid up its interior. In each of the boxes compute the total fluxes. Add all these fluxes. Since any net (positive/negative) flux from one box is taken ~~(in/out)~~ (in/out) by neighbor boxes, their total sum would zero out, EXCEPT for boxes near the surface S. Hence

Sum fluxes = Sum of fluxes of outer boxes, the shaded ones.

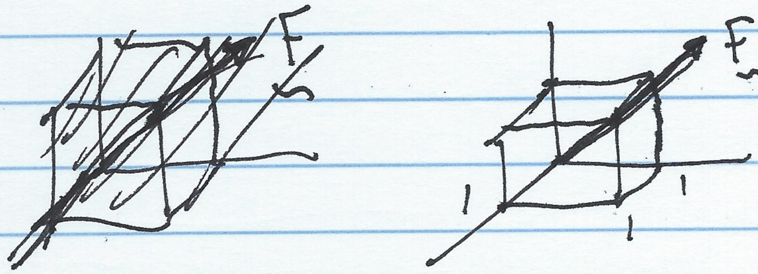
We saw that each box i contributes $(\nabla \cdot \underset{\sim}{q})_i \, dV_i$ fluxes so

sum of these must equal to the net fluxes leaving the volume //

ex) Verify the divergence theorem

$$\underset{\sim}{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

for a surface of a cube bounded by the coordinate planes and the planes $x=1$, $y=1$, $z=1$:

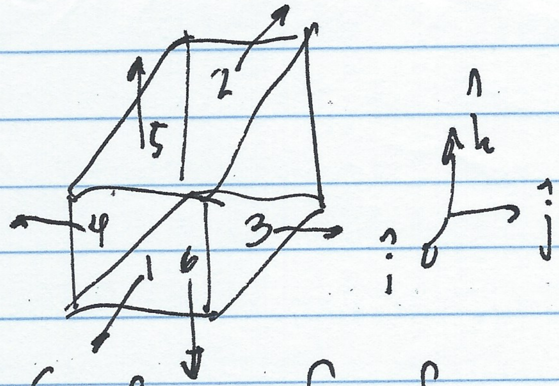


V has volume of 1.

$$\int_V \nabla \cdot \underset{\sim}{F} \, dV = \oint_S F \cdot \hat{n} \, dS$$

$\nabla \cdot \underset{\sim}{F} = \nabla \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3 \quad \therefore \int_V \nabla \cdot F \, dV = 3 \int_V dV = 3$

Now $\oint_S \underset{\sim}{F} \cdot \hat{n} \, dS = \int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{S_4} + \int_{S_5} + \int_{S_6}$



$$\oint_S \underset{\sim}{F}\cdot\hat{n}\,dS = \underbrace{\int_{S_1} + \int_{S_2}} + \underbrace{\int_{S_3} + \int_{S_4}} + \underbrace{\int_{S_5} + \int_{S_6}}$$

$$\int_{S_1+S_2} = \iint \underset{\sim}{F}\Big|_{S_1}\cdot\hat{i}\,dydz + \iint \underset{\sim}{F}\Big|_{S_2}\cdot(-\hat{i})\,dydz = \iint x\Big|_1 dydz - \iint 0\,dydz = 1$$

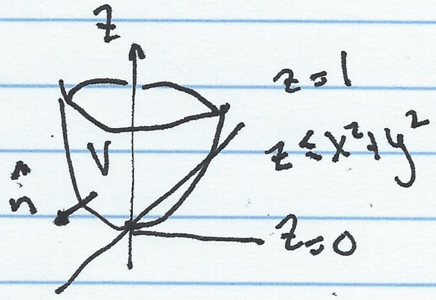
$$\int_{S_3+S_4} = \iint y\Big|_{y=1} dxdz - \iint 0\,dxdz = 1$$

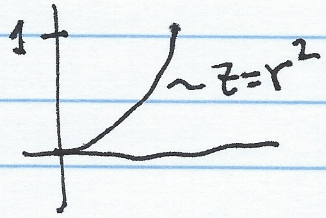
$$\int_{S_5+S_6} = 1$$

$$\therefore \oint_S \underset{\sim}{F}\cdot\hat{n}\,dS = 3 \qquad //$$

ex) Verify the divergence theorem

$$\underset{\sim}{E} = \frac{1}{2} y\,\hat{i} + \frac{1}{2} x\,\hat{j} + z^2 \hat{k}$$





$$\int_V (\nabla \cdot \underset{\sim}{E})\, dV = \oint_S \underset{\sim}{E} \cdot d\underset{\sim}{S}$$

$$\hookrightarrow \nabla \cdot \underset{\sim}{E} = 2z$$

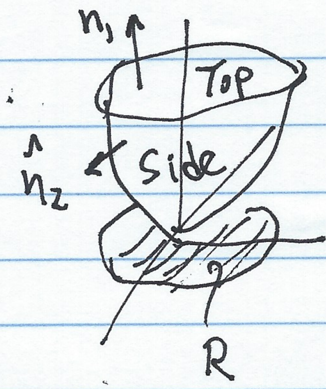
$$\iiint_V 2z\, dx\, dy\, dz = 2 \int_0^{2\pi} d\theta \int_0^1 dr\, r \int_{r^2}^1 z\, dz = 4\pi \int_0^1 dr\, r\, \frac{1}{2} z^2 \Big|_{r^2}^{1}$$

$$= \frac{4\pi}{2} \int_0^1 dr\, r\,(1 - r^4) = 2\pi \int_0^1 dr\,(r - r^5) = 2\pi \left(\frac{1}{2} r^2 - \frac{1}{6} r^6\right|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{6}\right) = \frac{2\pi}{3}.$$

Now, let's do

$$\oint_S \underset{\sim}{E} \cdot d\underset{\sim}{S} = \int_{\text{Top}} + \int_{\text{Side}}$$



Top: $\int_{Top} \underset{\sim}{E} \cdot \hat{n}_1 dS_1$

$\hat{n}_1 = \hat{k} \qquad dS_1 = dx\,dy$

$$\iint_{Top} \underset{1}{\underset{\shortparallel}{z^2}}\, dx\,dy = \iint_{Top} dx\,dy = \pi$$

Side: let $g = -z + x^2 + y^2 \qquad \hat{n}_2 = \frac{\nabla g}{|\nabla g|} = \frac{-\hat{k} + 2x\hat{i} + 2y\hat{j}}{\sqrt{1 + 2^2x^2 + 2^2y^2}}$

$$\int_{side} \underset{\sim}{E} \cdot \hat{n}_2 dS_2 = \iint_R \underset{\sim}{E} \cdot \hat{n}_2 \frac{dx\,dy}{|\hat{n}_2 \cdot \hat{k}|} = \iint_R \left(\tfrac{1}{2}y\hat{i} + \tfrac{1}{2}x\hat{j} + z^2\hat{k}\right) \cdot \hat{n}_2 \frac{dx\,dy}{|\hat{n}_2 \cdot \hat{k}|}$$

$$\hat{n}_2 \cdot \hat{k} = \frac{-1}{|\nabla g|} \quad \therefore \quad \frac{\hat{n}_2}{|\hat{n}_2 \cdot \hat{k}|} = |\nabla g|\,\hat{n}_2 = |\nabla g| \frac{\nabla g}{|\nabla g|} = \nabla g$$

$$\iint_R \left(\tfrac{1}{2}y\hat{i} + \tfrac{1}{2}x\hat{j} + z^2\hat{k}\right) \cdot \left(2x\hat{i} + 2y\hat{j} - \hat{k}\right) dx\,dy = \iint_R \left(xy + xy - z^2\right) dx\,dy$$

but $z = r^2 \therefore 2r^2 \sin\theta\cos\theta - r^4 = 2xy - z^2$

$$\int_0^{2\pi} d\theta \int_0^1 r\,dr\left(2r^2\sin\theta\cos\theta - r^4\right) = \int_0^{2\pi} d\theta \left[\tfrac{2}{3}r^3 \sin\theta\cos\theta - \tfrac{1}{6}r^6\right]_0^1$$

$$= \int_0^{2\pi} d\theta \left[\tfrac{2}{3}\sin\theta\cos\theta - \tfrac{1}{6}\right] = \int_0^{2\pi} \tfrac{1}{3}\frac{d}{d\theta}\left[\sin^2\theta\right] d\theta - \tfrac{1}{6}2\pi = -\tfrac{2}{6}\pi$$

Note:

$$\boxed{= \frac{1}{3}\sin^2\theta \Big|_0^{2\pi} = 0}$$

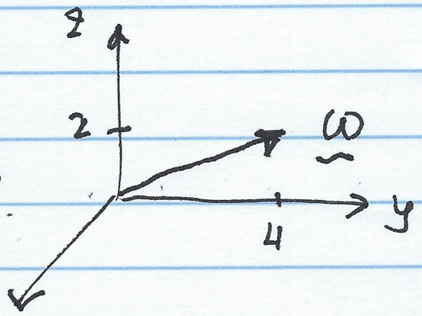
$$\therefore \int_{Top} + \int_{side} = \pi - \frac{1}{3}\pi = \frac{2\pi}{3}$$ which agrees with the $\int_V \nabla\cdot \underset{\sim}{E}\, dV$ result

ex) A rotating field $\underset{\sim}{V} = \langle 2z - y, x, -2x \rangle$

"rotating" because its curl is not zero:

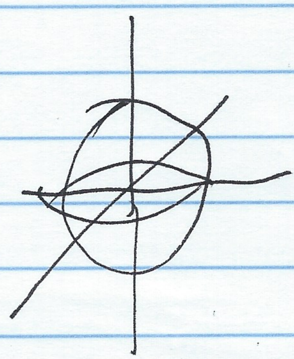
$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{V} = 4\hat{j} + 2\hat{k}$$

the rotation rate $|\underset{\sim}{\omega}| = \sqrt{4^2+2^2} = \sqrt{20}$



find total flux in a unit ball center at the origin.

$$\oint_S v\cdot\hat{n}\, dS = \int_V \nabla\cdot\underset{\sim}{v}\, dV$$



find $\nabla \cdot \underset{\sim}{v} = 0$ so total flux is 0. Done. But if you are wondering,

$$\oint_{\text{ball}} \underset{\sim}{v} \cdot \hat{n}\, dS \qquad \text{let } \underset{\sim}{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \hat{n} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|} = \hat{r}$$

$$\underset{\sim}{v} \cdot \hat{n}\Big|_{\substack{\text{ball}\\ \text{surface}}} = \frac{1}{|\underset{\sim}{r}|} \langle 2z-y, x, -2x \rangle \cdot (x\hat{i} + y\hat{j} + z\hat{k})\Big|_{x^2+y^2+z^2=1}$$

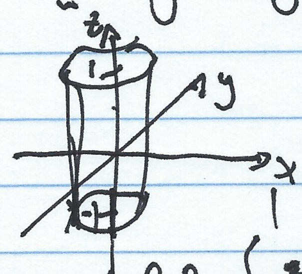
$$= \frac{1}{\sqrt{3}} \left[(2z-y)x + xy - 2xz \right]\Big|_{x^2+y^2+z^2=1} = 0$$

$$\oint_{\text{ball}} \underset{\sim}{v} \cdot \hat{n}\, dS = \oint_{\text{ball}} 0\, dS = 0$$

So far we've calculated $\oint_S \underset{\sim}{F} \cdot \hat{n}\, dS = \int_V \nabla \cdot \underset{\sim}{F}\, dV$ without concerning ourselves about what types of S & V's are OK for the above to hold. We also have not put any conditions on $\underset{\sim}{F}$. We consider that next.

ex) Verify $\int_V \nabla\cdot \underset{\sim}{F}\, dV = \oint_S \underset{\sim}{F}\cdot d\hat{S}$ where $\underset{\sim}{F} = xy^2\hat{i} + x^2y\hat{j} + y\hat{k}$

for a cylinder $-1 \le z \le 1$ $x^2+y^2=1$



$\nabla\cdot \hat{F} = x^2+y^2 = r^2$

$$\int_V (x^2+y^2)\,dx\,dy\,dz = \int_{-1}^{1} dz \int_0^{2\pi} d\theta \int_0^1 r\,dr\,r^2 = 2\cdot 2\pi \int_0^1 r^3\,dr = 2\cdot 2\pi \frac{1}{4} = \pi \quad //$$

$$\oint_S \hat{F}\cdot d\hat{S} = \int_{top} + \int_{bottom} + \int_{side}$$

$$\int_{top} + \int_{bottom} = \int_{S_{top}} \underset{\sim}{F}\cdot \hat{n}_{top}\,dx\,dy + \int_{S_{bottom}} \underset{\sim}{F}\cdot\hat{n}_{bottom}\,dx\,dy = \int_{S_{top}} y\hat{k}\cdot\hat{k}\,dx\,dy + \int_{S_{bottom}} y\hat{k}(-\hat{k})\,dx\,dy = 0$$

For the side $\hat{n} = \hat{r}$ and $dS = dz\,d\theta$; $F(x,y,z)$ needs to be expressed in cylindrical coordinates

$$\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{e}_\theta \qquad \hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{e}_\theta \qquad \hat{k} = \hat{k}$$

Since $F\cdot\hat{r}\,dS$ only has a contribution from $\hat{r}$, ignore all terms in the $\hat{e}_\theta$ and $\hat{k}$ direction

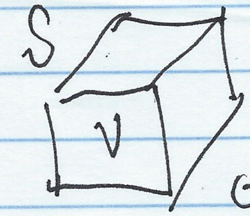
$$\underset{\sim}{F} = r^3\cos\theta\sin^2\theta(\cos\theta\,\hat{r} - \sin\theta\,\hat{e}_\theta) + r^3\cos^2\theta\sin\theta(\sin\theta\,\hat{r} + \cos\theta\,\hat{e}_\theta) + r\sin\theta\,\hat{k}$$

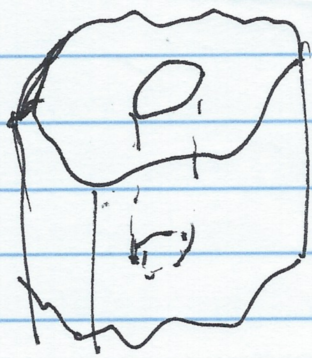
$$\underset{\sim}{F}\cdot\hat{r} = r^3\cos^2\theta\sin^2\theta + r^3\cos^2\theta\sin^2\theta = 2r^3\cos^2\theta\sin^2\theta$$
$$= \frac{1}{2}r^3(1+\cos 2\theta)(1-\cos 2\theta) = \frac{1}{2}r^3\sin^2 2\theta$$

$$\therefore \int_{side} \underset{\sim}{F}\cdot\hat{n}\,dS = \int_{-1}^{1} dz \int_0^{2\pi} \frac{1}{2}r^3\Big|_{r=1} \sin^2 2\theta\,d\theta = \int_0^{2\pi} \sin^2 2\theta\,d\theta$$

$$= \int_0^{2\pi} d\theta\,\frac{1}{2}(1-\cancelto{0}{\cos 4\theta}) = \int_0^{2\pi} d\theta\,\frac{1}{2} = \pi \quad //$$

S must be closed and continuous, So

ex)  ok (provided there's no openings

It could also be S, the top bottom, outer and inner surfaces & V is contained therein 

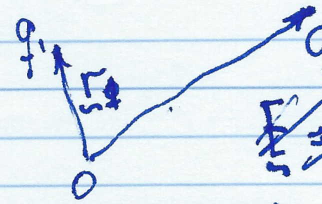
What about $\underset{\sim}{F}$? it must be continuous and its $\nabla \cdot \underset{\sim}{F}$ must be defined everywhere inside S. It must be defined on S.

So obviously, sometimes the choice of S, V is given by the application, sometimes by the region whereby $\underset{\sim}{F}$ has the right properties.

Let's see how this plays out. It leads to "Gauss' Law"

<u>Gauss' Law</u>

$\mathbf{F} = q_1 \mathbf{E}$



$\cancel{\mathbf{E} = k\,(\mathbf{r}-\mathbf{r}_1)}$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}$$

<u>Electricity</u>: ~~(Plate of infinity)~~

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

field due to a point charge (at origin)

ε_0 permitivity (of freespace)

Find the flux of this point charge at any surface enclosing it:

$$\oint_S \mathbf{E}\cdot\hat{n}\,d\sigma = \int_V \nabla\cdot\mathbf{E}\,dV$$

since $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$

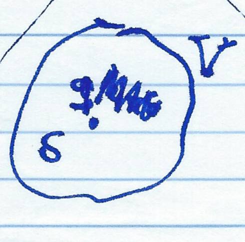
$$\nabla\cdot\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x}\left(\frac{x}{|\mathbf{r}|^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{|\mathbf{r}|^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{|\mathbf{r}|^3}\right)\right]$$

$$\frac{\partial}{\partial x}\left(x\left[x^2+y^2+z^2\right]^{-3/2}\right) = \left[x^2+y^2+z^2\right]^{-3/2} - \frac{3}{2}x\left[x^2+y^2+z^2\right]^{-5/2}(2x)$$

$$= |\mathbf{r}|^{-3} - 3x^2|\mathbf{r}|^{-5}$$

$$\therefore \nabla\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = 3|\mathbf{r}|^{-3} - 3(x^2+y^2+z^2)|\mathbf{r}|^{-5} = 0$$

$\therefore \oint_S \mathbf{E}\cdot\hat{n}\,dS = 0$ regardless of shape of S so long as it encloses V.



<u>Wrong!</u>

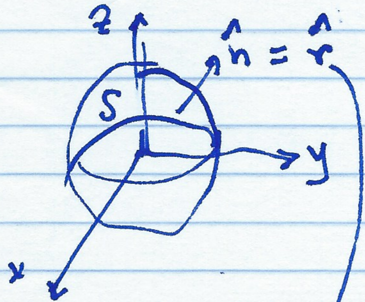
<u>Why??</u>

[Note that $\dfrac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}$ is not defined at origin!!

$\therefore\ \nabla \cdot \left(\dfrac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\right)$ is not defined either!]

New Calculation

Find the flux (we'll use a sphere)
the general case
is technically involved)



$$\oint_S \underset{\sim}{E} \cdot \hat{n} \, d\sigma$$

$$= \frac{Q}{4\pi\varepsilon_0} \int \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3} \cdot \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|} \, d\sigma = \frac{Q}{4\pi\varepsilon} \int \frac{|\underset{\sim}{r}|^2}{|\underset{\sim}{r}|^4} \, d\sigma = \frac{Q}{4\pi\varepsilon a^2} \overbrace{\int}^{\text{sphere}} d\sigma$$

$$= \frac{Q}{4\pi\varepsilon a^2} \, 4\pi a^2 = \frac{Q}{\varepsilon_0}$$

$$\text{so} \quad \oint_S \underset{\sim}{E} \cdot \hat{n} \, d\sigma = \frac{Q}{\varepsilon_0} = \overbrace{\frac{1}{\varepsilon_0} \int_V \rho \, dV}^{\text{Generally.}}$$

Total charge

//

~~Let's Fi~~ So we have

$$\cancel{\int \nabla \cdot \left(\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\right) dV}$$

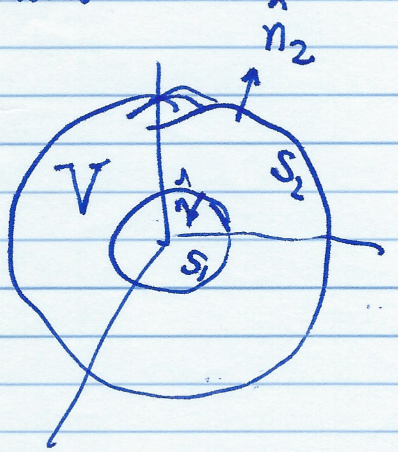
$$\nabla \cdot \left(\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\right) = 0 \quad \text{true if } \underset{\sim}{r} \neq \underset{-}{0}$$ (exclude the origin)

&

$$\int_V \nabla \cdot \underset{\sim}{E}\, dV = \oint_S \underset{-}{E} \cdot \hat{n}\, d\sigma = \frac{Q}{\varepsilon_0}$$

this can't be 0 and Q/ε_0 at the same time!

The solution:



Now

$$\nabla \cdot \left(\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\right) = 0$$

since the origin is not included

$$\int_V \nabla \cdot \left(\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\right) dV = 0$$

$$\frac{Q}{4\pi\varepsilon_0} \int_V \nabla \cdot \left(\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\right) dV = 0 = \int_{S_1} \underset{\sim}{E} \cdot \hat{n}_1\, d\sigma + \int_{S_2} \underset{\sim}{E} \cdot \hat{n}_2\, d\sigma$$

but $$\int_{S_2} \underset{\sim}{E} \cdot \hat{n}_2\, d\sigma = \frac{Q}{\varepsilon_0} \Rightarrow \int_{S_1} \underset{\sim}{E} \cdot \hat{n}_1\, d\sigma = -\frac{Q}{\varepsilon_0}$$