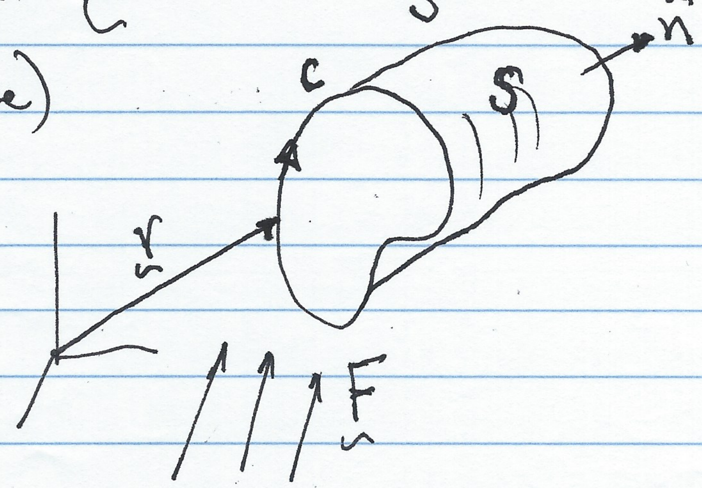
[illegible].7 Stokes' Theorem

$$\oint_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_S (\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot d\underset{\sim}{S}$$

"the circulation" (traditional name)



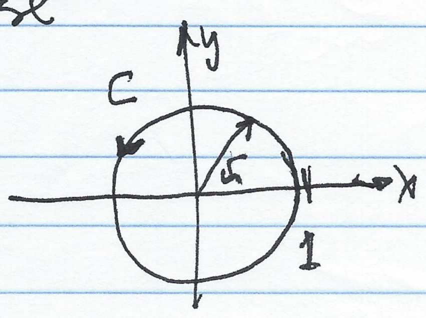
$d\underset{\sim}{S} = \hat{n}\,dS$

S is the surface enclosed by the closed curve C

$\hat{n}$ is the "right-hand" normal: curl your right hand fingers in the direction of the path C, the normal points in the direction of the extended thumb.

ex) Verify Stokes' Theorem: $\underset{\sim}{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x\hat{j}}{x^2+y^2}$

The path is a circle of radius 1, traversed counterclockwise



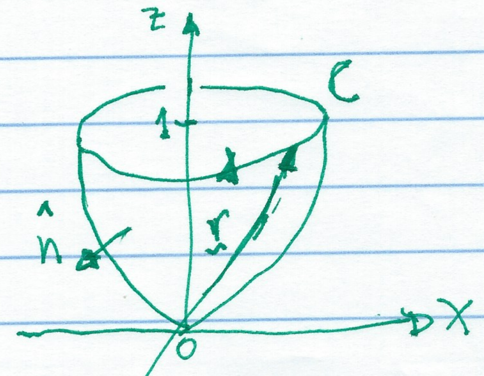
Try this!

II. ex) Verify Stokes' Theorem $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_S (\nabla \times \underset{\sim}{F}) \cdot d\underset{\sim}{S}$

$\underset{\sim}{F} = y^2 \hat{i} + x \hat{j} + z^2 \hat{k}$

(1) $S_1 = \{ z = x^2 + y^2 \ ; \ 0 \le z \le 1 \}$

(2) $S_2 = \{ 1 = x^2 + y^2 \}$



We expect that using either S_1 or S_2 yields the same answer as well as the outcome of the line integral.

(I) Do the line integral (note the direction of the path)

Let $\underset{\sim}{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j} + \hat{k}$

$\underset{\sim}{v}(t) = -\sin t\, \hat{i} + \cos t\, \hat{j}$

$\underset{\sim}{F}(t) = \sin^2 t\, \hat{i} + \cos t\, \hat{j} + z^2 \hat{k}$

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \underset{\sim}{F}(t) \cdot \underset{\sim}{v}\, dt = -\int_0^{2\pi} dt \left[-\cancel{\sin^3 t}^{\,0} + \cos^2 t \right]$$

$$= -\int_0^{2\pi} \frac{1}{2}(1 + \cancel{\cos 2t}^{\,0})\, dt = -\pi$$

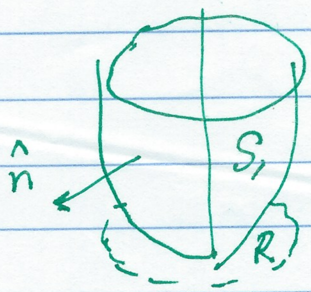
Using S_1: we let $g = x^2+y^2-z = 0$; Use $\int_{S_1} \nabla\times\underset{\sim}{F}\cdot d\underset{\sim}{S}$

then $\hat{n} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{2x\hat{i}+2y\hat{j}-\hat{k}}{\sqrt{4x^2+4y^2+1}}$

$$\nabla\times\underset{\sim}{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & x & z^2 \end{vmatrix} = +\hat{k}(1-2y)$$

$\cancel{\hat{n}\cdot\hat{k}}$ $\quad d\underset{\sim}{S} = \hat{n}\,dS$

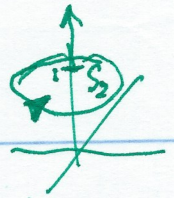
$$(\nabla\times\underset{\sim}{F})\cdot\hat{n}\,dS = -(1-2y)\,\hat{k}\cdot\hat{n}\,dS = \frac{-(1-2y)\,dS}{\sqrt{4x^2+4y^2+1}} \;\therefore\; -\int_{S_1}\frac{(1-2y)\,dS}{\sqrt{2x^2+2y^2+1}}$$

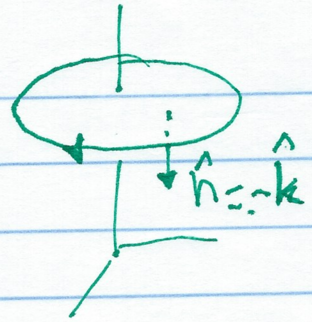


$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$-\int_{S_1}\cdot\,dS = -\iint_R \frac{(1-2y)}{|\hat{n}\cdot\hat{k}|}\,\hat{n}\cdot\hat{k}\,dx\,dy = -\iint_R (1-2y)\,dx\,dy$$

$$= -\int_{-1}^{1}dx\int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} dy\,(1-2y) = -\int_{-1}^{1}dx\,\left[y - y^2\right]_{y=-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} = -\pi \quad //$$





Now, try S_2:

$$\int_{S_2} (\nabla \times \underset{\sim}{F}) \cdot d\underset{\sim}{S}$$

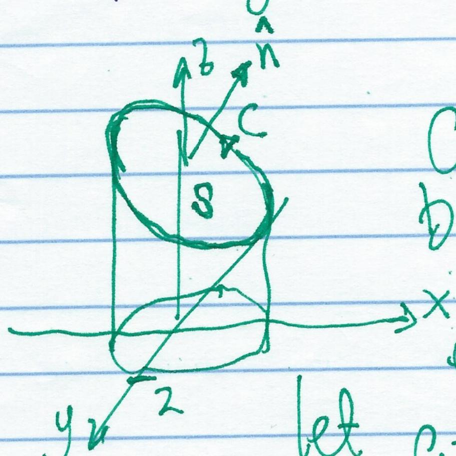
$$d\underset{\sim}{S} = -\hat{k}\, dx\, dy$$

$$\nabla \times \underset{\sim}{F} = \hat{k}(1-2y)$$

$$-\int_{S_2} (1-2y)\, dx\, dy = -\int_{-1}^{1} dx \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} dy\, (1-2y) = -\pi$$

ex) Use Stokes' Theorem to find $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$\underset{\sim}{F} = \langle -y, x^2, z^2 \rangle$, C is the intersection of $x^2+y^2=4$ and $x+z=3$ plane



C is not difficult to parametrize but the surface integral involving S might be easy to ~~evalute~~ evaluate.

let $g = x+z-3 = 0$

then $\hat{n} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle$

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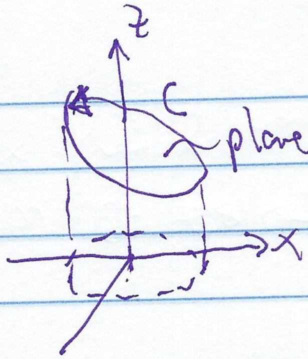
The line integral



$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \oint_C \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}\, dt$$

$$\underset{\sim}{r} = \langle 2\cos t, 2\sin t, z(t) \rangle$$

on the plane $z(t) = 3 - x(t) = 3 - 2\cos t$

$$\underset{\sim}{r} = \langle 2\cos t, 2\sin t, 3 - 2\cos t \rangle \qquad 0 \le t \le 2\pi$$

$$\frac{d\underset{\sim}{r}}{dt} = \langle -2\sin t, 2\cos t, +2\sin t \rangle$$

$$\underset{\sim}{F} = \langle y, x^2, z^2 \rangle = \langle -2\sin t, 2^2\cos^2 t, (3-2\cos t)^2 \rangle$$

$$\underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} = 4\sin^2 t + 8\cos^3 t + 2\sin t(9 - 12\cos t + 4\cos^2 t)$$

$$\int_0^{2\pi} \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt}\, dt = \int_0^{2\pi} [4\sin^2 t + \cancel{8\cos^3 t}^{\,0} + \cancel{18\sin t}^{\,0} - \cancel{24\sin t\cos t}^{\,0} + \cancel{8\sin t\cos^2 t}^{\,0}]\, dt$$

$$= \int_0^{2\pi} 4\sin^2 t\, dt = 2\int_0^{2\pi} (1 + \cancel{\cos 2t}^{\,0})\, dt = 2\int_0^{2\pi} dt = 4\pi$$

Now, as a surface integral:

$$\nabla\times F = (2x+1)\hat{k} \quad \text{and} \quad dS = \hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

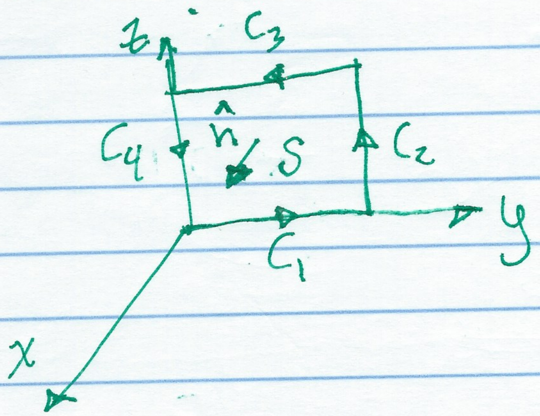
$$\int_S (\nabla\times F)\cdot dS = \iint_R (2x+1)\,\frac{\hat{k}\cdot\hat{n}\,dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

$$= \int_R dy\int dx\,(2x+1) = \int_0^{2\pi} d\theta \int_0^2 r\,dr\,(2r\cos\theta+1)$$

$$= \int_0^2 r\,dr\left(\overset{0}{\cancel{-2r\sin\theta\Big|_0^{2\pi}}} + \theta\Big|_0^{2\pi}\right) = 2\pi\int_0^2 r\,dr = \frac{2\pi}{2}r^2\Big|_0^2 = \pi 4$$

ex) Verify Stokes Theorem, $F = \hat{i} + zy^2\hat{j}$

on a flat surface bounded by $(0,0,0)$, $(0,1,0)$, $(0,1,1)$, $(0,0,1)$



$$\oint_C F\cdot dr = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$= \int_S (\nabla\times F)\cdot dS = \int_S (\nabla\times F)\cdot\hat{i}\,dy\,dz$$

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$$\oint_C = \int_{C_1} + \cdots + \int_{C_4}$$

$$\underline{F} = \hat{i} + zy^2\hat{j}$$

let $\underline{r} = y\hat{j} + z\hat{k}$

$d\underline{r} = dy\,\hat{j} + dz\,\hat{k}$

C_1:

$$\int_{C_1} \underline{F}\cdot d\underline{r} = \int_0^1 \overset{0}{z}\, y^2\, dy = 0$$

C_2:

$$\int_{C_2} \underline{F}\cdot d\underline{r} = \int_0^1 0\, dz = 0$$

C_3:

$$\int_{C_3} \underline{F}\cdot d\underline{r} = \int_1^0 \overset{1}{z}\, y^2\, dy = \int_1^0 y^2\, dy = -\int_0^1 dy\, y^2 = -\frac{1}{3} y^3 \Big|_0^1 = -\frac{1}{3}$$

C_4

$$\int_{C_4} \underline{F}\cdot d\underline{r} = \int_0^0 \overset{0}{z}\, y^2\, dy = 0 \qquad \therefore \oint_C = -\frac{1}{3}$$

The Surface integral:

$$\nabla\times\underline{F} = -\frac{\partial}{\partial z}(zy^2)\,\hat{i} = \cancel{-zy\,\hat{i}}\; - y^2\,\hat{i}$$

$$\nabla\times\underline{F}\cdot\hat{n}\, dy\, dz = -y^2\, dy\, dz$$

$$\int_S (\nabla\times F)\cdot\hat{n}\, dy\, dz = -\int_0^1\int_0^1 y^2\, dy\, dz = -\int_0^1 y^2\, dy = -\frac{1}{3} \qquad //$$

Just as there might exist a scalar potential ϕ such that $\underset{\sim}{F} = \nabla\phi$ (i.e. if $\nabla \times \underset{\sim}{F} = 0$) there is a vector potential associated with certain vector fields.

Before showing this, let's note the following about a scalar potential and a conservative field:

Take a scalar potential ϕ and define another one $\Phi = \phi + K$, where K is a constant. Then

$\nabla\phi = \nabla\Phi = \nabla(\phi + K) \Rightarrow$ in the absence of further constraints the scalar potential that defines a vector field $\underset{\sim}{F}$ is not unique: $\underset{\sim}{F} = \nabla\Phi = \nabla(\phi + K) = \nabla\phi$.

In physics we say that we choose a "Gauge" when we choose or pin down the value of K via some constraint in the physics.

The Vector Potential

Suppose a vector $\underset{\sim}{B}$ has the property that

$\nabla \cdot \underset{\sim}{B} = 0$

We recall the identity $\nabla \cdot (\nabla \times \underset{\sim}{F}) = 0$ for any $\underset{\sim}{F}$ that's twice differentiable.

So we could set $\nabla \cdot \underset{\sim}{B} = \nabla \cdot (\nabla \times \underset{\sim}{A}) = 0$

We call $\underset{\sim}{A}$ the vector potential. We say that any vector $\underset{\sim}{B}$ that has zero divergence can be associated with a vector potential $\underset{\sim}{A}$, such that $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$ (given some differentiability requirements).

Given $\underset{\sim}{B}$ how do we find $\underset{\sim}{A}$?

Let $\underset{\sim}{B} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ be known.

Let $\underset{\sim}{A} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ be unknown.

by definition $\nabla \times \underset{\sim}{A} = \underset{\sim}{B}$ means that

$$
(*) \quad \begin{cases}
\dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z} = b_1 \\[2ex]
\dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x} = b_2 \\[2ex]
\dfrac{\partial a_3}{\partial x} - \dfrac{\partial a_1}{\partial y} = b_3
\end{cases}
$$

3 equations for 3 unknowns a_1, a_2, a_3.

The procedure is to solving these via integrations. Sometimes these integrals are difficult, but even if we succeed, we'll find that the resulting A is <u>not</u> uniquely defined <u>unless</u> ~~the~~ there are extra constraints imposed.

We will find that $\bar{\underset{\sim}{A}} = \underset{\sim}{A} + \nabla\psi$ is also a vector potential of $\underset{\sim}{B}$ if $\nabla \cdot \underset{\sim}{B} = 0$ why?

$$\nabla \times \bar{\underset{\sim}{A}} = \nabla \times (\underset{\sim}{A} + \nabla\psi) = \nabla \times \underset{\sim}{A} \quad \text{since } \nabla \times \nabla\psi = 0 \text{ always.}$$

For simple cases we can find $\underset{\sim}{A}$:

ex) Let $\underset{\sim}{B} = B_3 \hat{k}$, where ~~$B_3$~~ B_3 is a constant.

Clearly $\nabla \cdot \underset{\sim}{B} = 0$. Let's use (*):

① $\dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z} = 0$ ② $\dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x} = 0$

③ $\dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y} = B_3$

a little trial and error helps: assume $a_1 = 0$, i.e.

$\underset{\sim}{A} = a_2 \hat{j} + a_3 \hat{k}$

by ② $-\dfrac{\partial a_3}{\partial x} = 0$ and by ③ $\dfrac{\partial a_2}{\partial x} = B_3$

$\hookrightarrow a_3 = a_3(y, z)$

by ① $\dfrac{\partial a_3}{\partial y} = \dfrac{\partial a_2}{\partial z}$ if we assume $a_3 = a_3(z)$

$\therefore \dfrac{\partial a_2}{\partial z} = 0 \;\therefore a_2 = a_2(x, y)$

by ③ $\dfrac{\partial a_2(x,y)}{\partial x} = B_3 \Rightarrow a_2(x,y) = B_3 x + f(y)$

we could set $f(y) = 0$ and still get consistency

$\therefore \underset{\sim}{A} = \hat{j} x B_3$.

We ~~could~~ ~~have~~ chosen $a_3 = 0$

then $\frac{\partial a_2}{\partial z} = 0 \qquad \frac{\partial a_1}{\partial z} = 0$

$a_2 = a_2(x,y) \qquad a_1 = a_1(x,y)$

by (3), if $a_2 = a_2(y)$

$$-\frac{\partial a_1}{\partial y} = B$$

$$a_1 = -By + f(x)$$

set $f(x) = 0$

$$\Rightarrow \underset{\sim}{A} = -By\,\hat{i}$$

for either case $\underset{\sim}{A}_1 = \underset{\sim}{A} + \nabla\psi$ is also a vector potential to $\underset{\sim}{B} = B_3\hat{k}$

Choosing or constraining $\nabla\psi$ means choosing a "Gauge"

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