

14.6 Surface Integrals

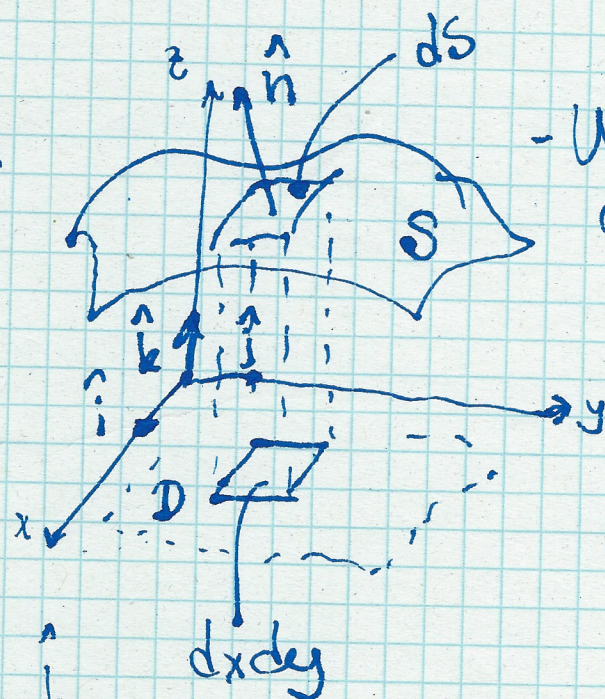
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We will focus on 2 (related) methods

(A) Projection

(B) (Parametrization) JACOBIAN TRANSFORMATION

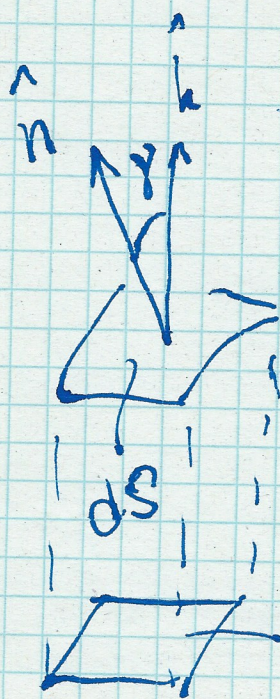
(A) Projection



- Want to find surface of S , described by

$$z = f(x, y)$$

for some specified range in (x, y) .



$$\hat{n} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

(by geometry)

$$|\hat{n} \cdot \hat{k}| dS = dxdy$$

The total area $A = \int dS$

let $g(x, y, z) = z - f(x, y) = 0$

① find $\hat{n} = \frac{\nabla g}{|\nabla g|} = \frac{-\frac{\partial f}{\partial x} \hat{i} - \frac{\partial f}{\partial y} \hat{j} + \hat{k}}{\sqrt{1 + f_x^2 + f_y^2}}$

$$|\hat{n} \cdot \hat{k}| = \cos \gamma = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} = \frac{1}{\sec \gamma}$$

$$dS = \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$$

$$A = \int_S dS = \iint_D \frac{dx dy}{|\hat{n} \cdot \hat{k}|} = \iint_D |\sec \gamma| dx dy$$

the projection of S into x, y plane

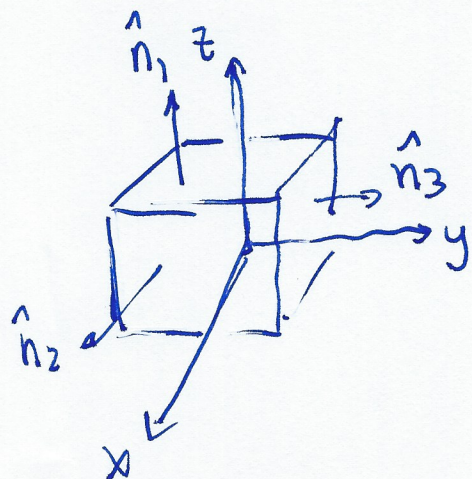
$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2} dx dy$$

ex) Find outward/inward normal to unit sphere: 65

let $g_+ = x^2 + y^2 + z^2 - 1 = 0$ (outward) and $g_- = -x^2 - y^2 - z^2 + 1 = 0$ (inward)

$$\hat{n}_{\pm} = \frac{\nabla g_{\pm}}{|\nabla g_{\pm}|} = \pm \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \pm \hat{r}$$

ex) Find outward normal to unit box: do this one by inspection:

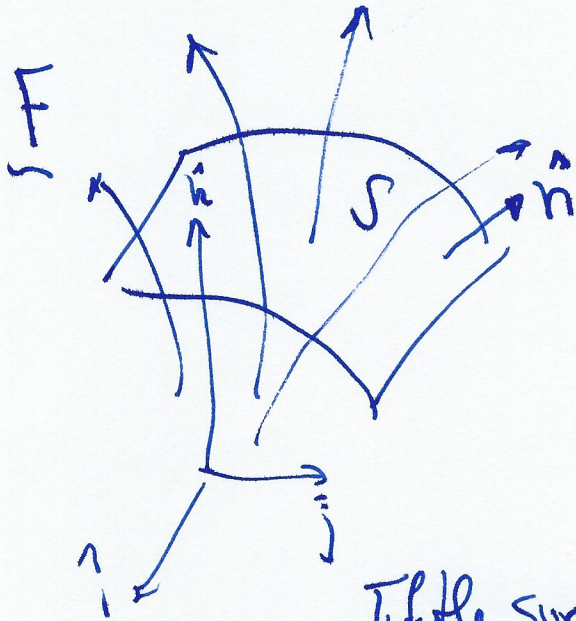


$$\begin{aligned}\hat{n}_1 &= \hat{k} \\ \hat{n}_2 &= \hat{i} \\ \hat{n}_3 &= \hat{j}\end{aligned}$$

$$\begin{aligned}n_4 &= -\hat{k} \\ n_5 &= -\hat{i} \\ n_6 &= -\hat{j}\end{aligned}$$

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An application of surface integrals is the calculation of the "flux" across a surface S



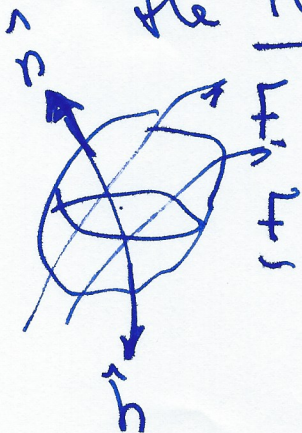
The flux is a scalar

$$\int_S \vec{F} \cdot d\vec{S}$$

where $d\vec{S} = \hat{n} dS$

If the surface S is closed we might want

the Total Flux



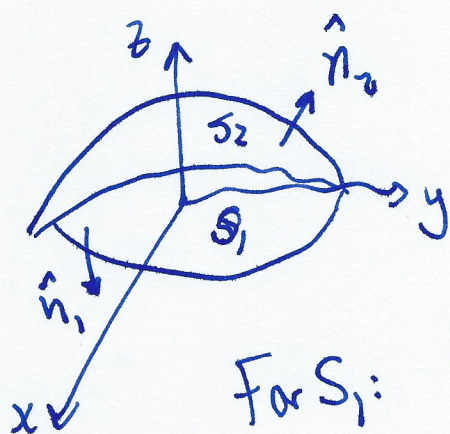
"means
"closed"

$$\oint_S \vec{F} \cdot d\vec{S}$$

ex) Find total flux, if $\underline{F} = x\hat{i} + y\hat{j}$

$S: x^2 + y^2 + z^2 = 1$ and the bottom cap

at $z=0$



$$\oint_S \underline{F} \cdot d\underline{S} = \int_{S_1} \underline{F} \cdot d\underline{S}_1 + \int_{S_2} \underline{F} \cdot d\underline{S}_2$$

For S_1 : $\hat{n}_1 = -\hat{k} \therefore d\underline{S}_1 = -\hat{k} dx dy$

but $\underline{F} \cdot d\underline{S}_1 = 0 \therefore \int_{S_1} = 0$.

Now focus on S_2 : $\hat{n}_2 = x\hat{i} + y\hat{j} + z\hat{k}$. The projection of S_2 onto x - y plane is a unit circle $x^2 + y^2 = 1$.

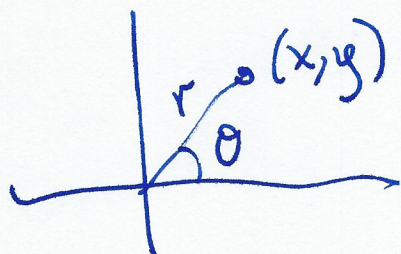
$$\left\{ \begin{array}{l} \hat{n}_2 \cdot \hat{k} = z = \cos \theta \\ d\underline{S}_2 = \hat{n}_2 dS = \hat{n}_2 \frac{dx dy}{z} \end{array} \right\} \therefore \int_{S_2} \underline{F} \cdot d\underline{S}_2 = \int_D (x^2 + y^2) \frac{dx dy}{z}$$

$$\text{Flux} = \int_D (x^2 + y^2) \frac{dx dy}{\sqrt{1 - (x^2 + y^2)}} = 4 \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \frac{x^2 + y^2}{\sqrt{1 - (x^2 + y^2)}}$$

We can exploit symmetries

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we note that

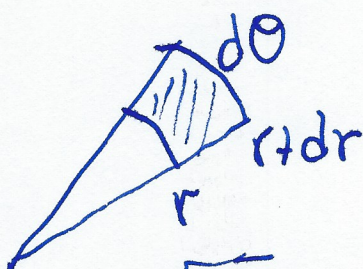


$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x}$$



shaded area has area
 $dr (r d\theta) = r dr d\theta$

$$\text{flux} = 4 \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \frac{x^2 + y^2}{\sqrt{1 - (x^2 + y^2)}} = 4 \int_0^{\pi/2} d\theta \int_0^1 \frac{r^2}{\sqrt{1-r^2}} r dr$$

$$= \frac{4\pi}{2} \int_0^1 \frac{r^3}{\sqrt{1-r^2}} dr = 2\pi \left[-\frac{1}{2} r^2 - \frac{1}{2} \ln(r-1) - \frac{1}{2} \ln(r+1) \right]_0^1$$

$$= 2\pi \int_0^1 \frac{r^3 dr}{\sqrt{1-r^2}} = 2\pi I$$

$$I = \int_0^1 \frac{r^3 dr}{\sqrt{1-r^2}} = \int_0^1 \frac{u du}{(1-u)^{1/2}} \quad \text{then}$$

let $u = r^2$
 $du = 2r dr$

integrate
by parts

$$S = u \quad w = -2(1-u)^{1/2}$$

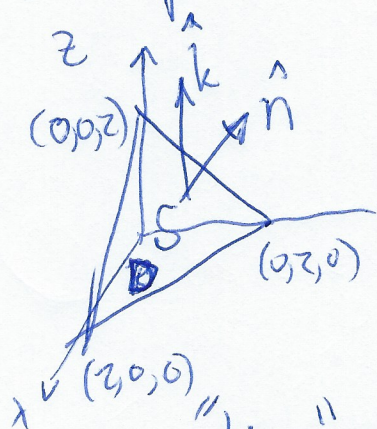
$$ds = du \quad dw = (1-u)^{-1/2} du$$

$$I = -u(1-u)^{1/2} \Big|_0^1 + \int_0^1 (1-u)^{1/2} du = - \int_1^0 p dp = \frac{2}{3}$$

$$\begin{cases} p = 1-u \\ dp = -du \end{cases}$$

$$\therefore \text{flux} = 2\pi \left(\frac{2}{3} \right) = 4\pi/3$$

ex) Find flux of $\underline{F} = x\hat{i} + y\hat{j} + z\hat{k}$
over the triangular region bounded by
the points $(2,0,0), (0,2,0), (0,0,2)$



① Find $\hat{n}$: clearly, the vector
 $\langle 2,0,0 \rangle - \langle 0,0,2 \rangle = \underline{u}_1 = \langle 2,0,-2 \rangle$

and

$$\langle 0,2,0 \rangle - \langle 0,0,2 \rangle = \underline{u}_2 = \langle 0,2,-2 \rangle$$

"line" on the surface and are not linearly
dependent. So $\hat{n} = \frac{\underline{u}_1 \times \underline{u}_2}{|\underline{u}_1 \times \underline{u}_2|} = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle$

$$\text{flux} = \int_S \underline{F} \cdot d\underline{S} \quad \text{where} \quad d\underline{S} = \hat{n} dS$$

$$dS = \frac{dx dy}{|\hat{n} \cdot \hat{k}|} = \sqrt{3} dx dy$$

$$\underline{F} \cdot \hat{n} = \frac{1}{\sqrt{3}} (x+y+z) \quad z = 2-x-y \text{ on } S$$

$$\begin{aligned} \text{flux} &= \int_D (x+y+z) dx dy = \iint (x+y+2-x-y) dx dy = \iint 2 dx dy \\ &= 2 \int_0^2 dx \int_0^{2-x} dy = 2 \int_0^2 dx (2-x) = 4 \end{aligned}$$

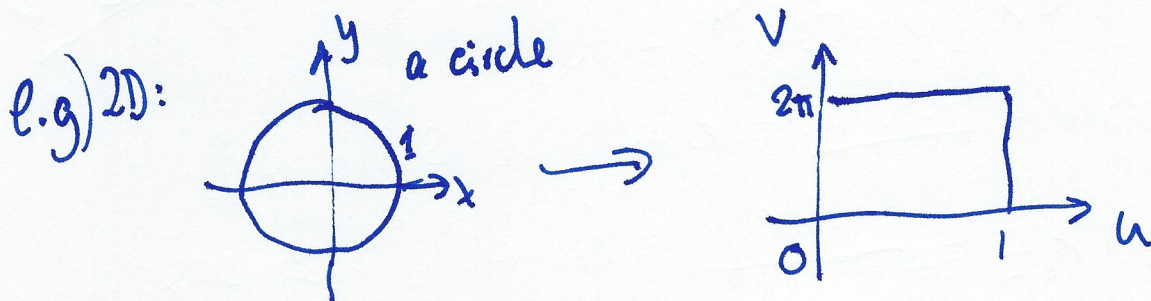
B Jacobian Transformation

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The idea is to transform

$$\iint_S f(x, y, z) dS \text{ into an integral}$$

Over a rectangular domain $R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$



so here $x(u, v)$ and $y(u, v)$

$$\text{are } x = u \cos v$$

$$y = u \sin v$$

$$u = \sqrt{x^2 + y^2}$$

$$\tan v = \frac{y}{x}$$

so we transform S to R and $dS = J du dv$

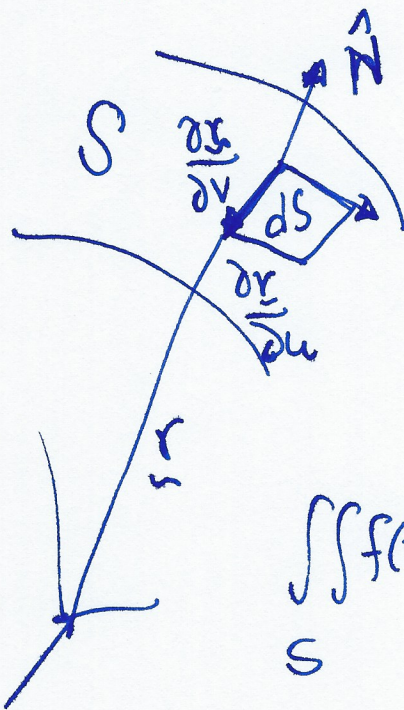
$$\text{and } f(x, y) = f(u, v)$$

J is a non-negative factor (see next)

if $\underline{r} = \underline{r}(x, y, z)$ can be parametrized by u, v

so that $\underline{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$

(usually u, v exploit geometric symmetries), then



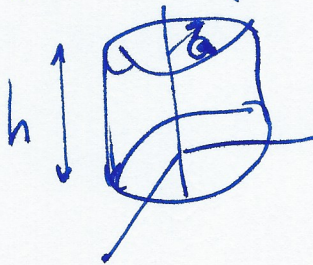
$$|\hat{N}| = |\hat{t}_u \times \hat{t}_v|$$

$$\hat{t}_u = \frac{\partial \underline{r}}{\partial u} \quad \text{and} \quad \hat{t}_v = \frac{\partial \underline{r}}{\partial v}$$

$$dS = |\hat{t}_u \times \hat{t}_v| du dv$$

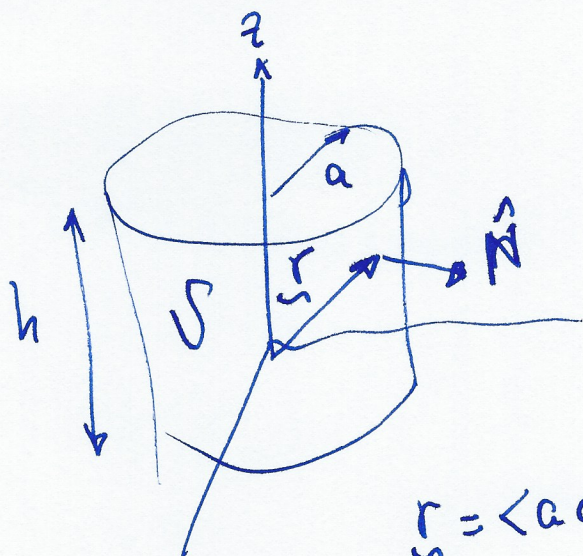
$$\iint_S f(x, y, z) dS = \int_c^d \int_a^b f(x(u, v), y(u, v), z(u, v)) |\hat{t}_u \times \hat{t}_v| du dv$$

ex) Compute the side surface area of a cylinder
height h



radius a

We know we should get $2\pi ah$



$$x(u, v) = a \cos u$$

$$y(u, v) = a \sin u$$

$$z(u, v) = v$$

$$\underline{r} = \langle a \cos u, a \sin u, v \rangle$$

$$\hat{N} = \underline{t}_u \times \underline{t}_v = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$\hat{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i} a \cos u + \hat{j} a \sin u + 0 \hat{k}$$

$$|\hat{N}| = \sqrt{a^2 \cos^2 u + a^2 \sin^2 u} = a$$

$$\therefore \iint_S f(x, y, z) dS = \int_0^h \int_0^{2\pi} |t_u \times t_v| du dv = a \int_0^h dv \int_0^{2\pi} du = 2\pi a h //$$

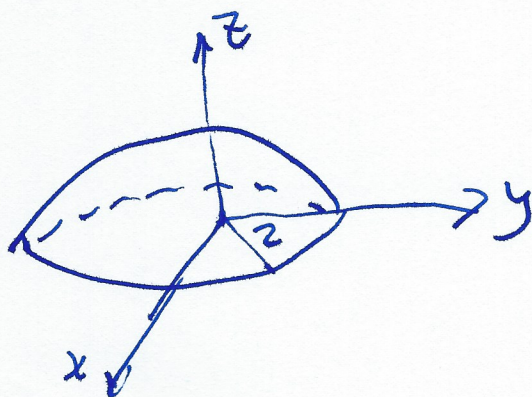
How to compute fluxes?

$$\iint_S \underline{F} \cdot \underline{\hat{n}} dS = \iint_R \underline{F} \cdot \underline{\hat{n}} |\underline{t}_u \times \underline{t}_v| du dv$$

$$\text{but } \underline{\hat{n}} = \frac{\underline{t}_u \times \underline{t}_v}{|\underline{t}_u \times \underline{t}_v|} = \frac{\underline{N}}{|\underline{N}|}$$

$$\therefore \iint_S \underline{F} \cdot \underline{\hat{n}} dS = \iint_R \underline{F} \cdot (\underline{t}_u \times \underline{t}_v) du dv$$

For comparison, let's calculate the total flux due to $\underline{F} = z \underline{\hat{k}}$ across a hemispheric cap $x^2 + y^2 + z^2 = 4$ $z \geq 0$ and the circle projection on the z plane



Total Flux

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$$\oint_S \underline{F} \cdot \hat{n} dS = \int_{\text{bottom}} \underline{F} \cdot \hat{n} dS + \int_{\text{cap}} \underline{F} \cdot \hat{n} dS$$

The bottom:

since $\underline{F} \cdot \hat{n} = z \hat{k} \cdot (-\hat{k}) = -z$ and $z = 0$ on bottom

$$\int_{\text{bottom}} = 0$$

The cap:

by (A) Projection

$$\hat{n} = \hat{r} = \frac{\underline{r}}{|\underline{r}|}$$

$$dS = \frac{1}{|\hat{n} \cdot \hat{k}|} dx dy$$

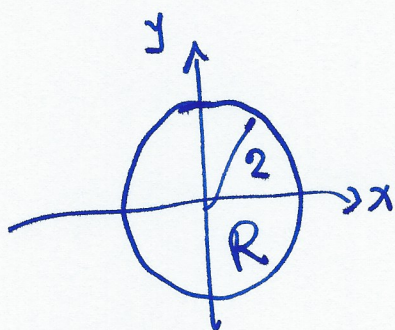
$$\int_{\text{cap}} \underline{F} \cdot \hat{n} dS = \int_R \underline{F} \cdot \hat{n} \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$$

$$\hat{n} \cdot \hat{k} = \cancel{\frac{\underline{r} \cdot \hat{k}}{|\underline{r}|}} \quad \hat{r} \cdot \hat{k} = \frac{z}{|\underline{r}|}$$

$$dS = \frac{|\underline{r}|}{z} dx dy$$

$$\underline{F} \cdot \hat{n} = z \hat{k} \cdot \frac{\underline{r}}{|\underline{r}|} = \frac{z^2}{|\underline{r}|}$$

$$\iint_R \frac{F \cdot \hat{n}}{|\hat{n} \cdot \underline{\hat{k}}|} dx dy = \iint_R \frac{z^2}{|\underline{r}|} \frac{|\underline{r}|}{z} dx dy = \iint_R z dx dy \quad 76$$



$$\text{but on } S \quad z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$$

$$r = \sqrt{x^2 + y^2}$$

$$\int_0^{2\pi} d\theta \int_0^2 dr \, r \sqrt{4 - r^2} = 2\pi \int_0^2 \sqrt{4 - r^2} r dr = \frac{2\pi}{-2} \int_4^0 u^{1/2} du$$

$$u = 4 - r^2 \quad r=0 \rightarrow u=4$$

$$du = -2r dr \quad r=2 \rightarrow u=0$$

$$= \pi \int_0^4 u^{1/2} du = \frac{2\pi}{3} u^{3/2} \Big|_0^4 = \frac{2\pi}{3} \cdot 8 = \frac{16\pi}{3} //$$

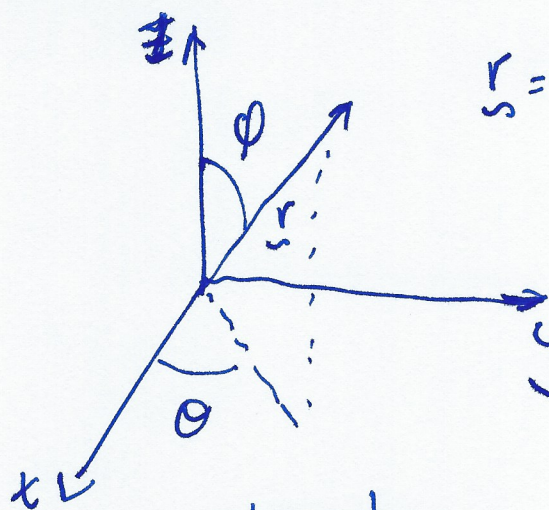
let's do the same using Jacobian Transformation (B)

$$\oint_S F \cdot \hat{n} ds = \int_{\text{bottom}} + \int_{\text{cap}} F \cdot \hat{n} ds$$

$$\int_{\text{cap}} F \cdot \hat{n} ds = \iint_R F \cdot (\underline{t}_u \times \underline{t}_v) du dv$$

we'll use $(u, v) = (\theta, \varphi)$

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$$\underline{r} = \langle 2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi \rangle$$

$$\underline{t}_\theta = \frac{\partial \underline{r}}{\partial \theta} = \langle -2 \sin \theta \sin \varphi, 2 \cos \theta \sin \varphi, 0 \rangle$$

$$\underline{t}_\varphi = \frac{\partial \underline{r}}{\partial \varphi} = \langle 2 \cos \theta \cos \varphi, 2 \sin \theta \cos \varphi, -2 \sin \varphi \rangle$$

$$\hat{n} = \frac{\underline{t}_\varphi \times \underline{t}_\theta}{|\underline{t}_\varphi \times \underline{t}_\theta|}$$

$$\underline{t}_\theta \times \underline{t}_\varphi = -2^2 \cos \varphi \sin^2 \varphi \hat{i} - 2^2 \sin \theta \sin^2 \varphi \hat{j} - 4 \sin \theta \cos \varphi \hat{k}$$

since $\underline{F} = z \hat{k}$ $\underline{F} \cdot (\underline{t}_\varphi \times \underline{t}_\theta) = -4 \sin \varphi \cos \varphi = 4 \cdot 2 \cos \varphi \sin \varphi \cos \varphi$
 $= 8 \cos^2 \varphi \sin \varphi$

$$\int_{\text{top}} \underline{F} \cdot (\underline{t}_\varphi \times \underline{t}_\theta) d\theta d\varphi = 8 \int \int_{\mathcal{R}(\theta, \varphi)} \cos^2 \varphi \sin \varphi d\theta d\varphi$$

$$= 8 \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \cos^2 \varphi \sin \varphi = 16\pi \int_0^{\pi/2} d\varphi \cos^2 \varphi \sin \varphi = 16\pi G$$

let $w = \cos \varphi$ $s = -\frac{1}{2} \cos^2 \varphi$

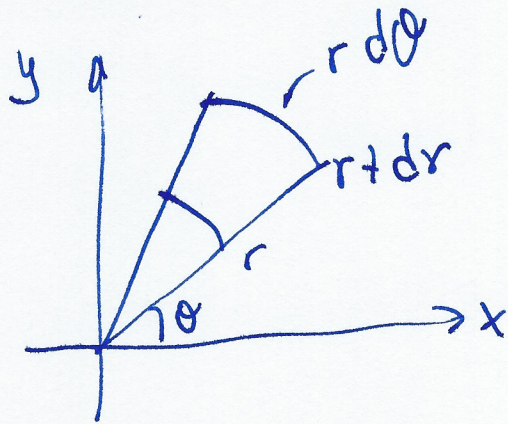
$dw = -\sin \varphi d\varphi$ $ds = \cos \varphi \sin \varphi d\varphi$

$$G = -\frac{1}{2} \cos^3 \varphi \Big|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \cos^3 \varphi \sin \varphi d\varphi = -\frac{1}{2} \cos^3 \varphi \Big|_0^{\pi/2} - \frac{1}{2} G \quad \therefore \frac{3}{2} G = -\frac{1}{2} \cos^3 \varphi \Big|_0^{\pi/2}$$

$$G = -\frac{1}{3} [\cos^3 \varphi]_0^{\pi/2} = \frac{1}{3} \quad \therefore \oint \underline{F} \cdot \hat{n} dS = \frac{16\pi}{3} //$$

Polar Differential Element 2D

Cartesian to Polar



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x}$$

$$x(r, \theta) = r \cos \theta$$

$$y(r, \theta) = r \sin \theta$$

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

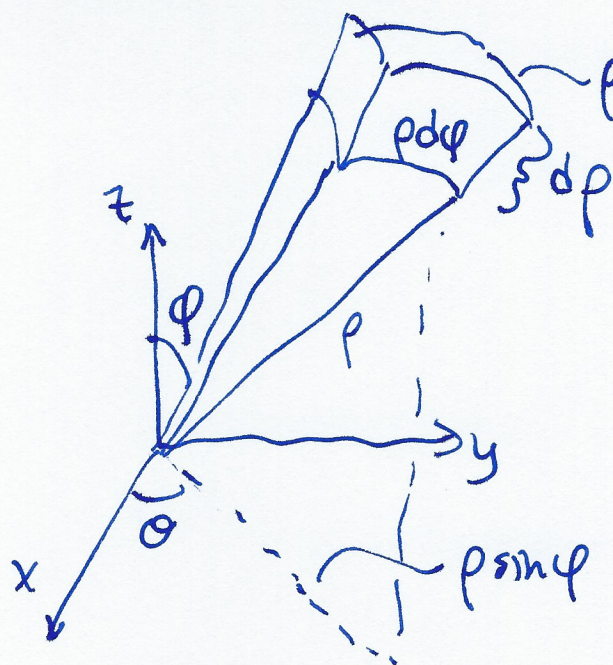
$$J = \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r}$$

$$J = \cos \theta \cdot r \sin \theta - (-r \sin \theta) \cos \theta = r$$

$$\therefore dx dy = |J| dr d\theta = r dr d\theta$$

Spherical Volume Element

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$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$x = \rho \cos \theta \sin \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \phi$$

$$dx \, dy \, dz = |J| \, d\rho \, d\theta \, d\phi$$

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \rho^2 \sin \phi$$

$$dx \, dy \, dz = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$