14.5 Divergence & Curl

$$\underset{\sim}{F} = f\hat{i} + g\hat{j} + h\hat{k}$$

$$\nabla \cdot \underset{\sim}{F} = \text{div}\,\underline{F} \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle$$

if $\nabla \cdot \underset{\sim}{F} = 0$ we say $\underset{\sim}{F}$ is source free
or incompressible

ex) $\underset{\sim}{F} = \langle x, y, z \rangle$ a radial field

$\nabla \cdot \underset{\sim}{F} = 3$

$\underset{\sim}{F} = \langle -y, z, x \rangle$ a rotative field

$\nabla \cdot \underline{F} = 0$

properties

$$\nabla \cdot (\nabla \times \underset{\sim}{F}) = 0$$

$$\nabla \cdot (u(x,y,z)\,\underset{\sim}{F}) = \nabla u \cdot \underset{\sim}{F} + u(\nabla \cdot F)$$

$$\nabla \cdot \nabla f = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$$

14.5 Div, Grad, Curl

Previously discussed the Grad (gradient) operator

in 3D $\quad \nabla f(\underset{\sim}{r}) = \langle \partial_x, \partial_y, \partial_z \rangle f(x,y,z)$

∇ when it operates on a scalar, ~~and~~ yields a vector.

We introduce 2 other "differential" operators:

The Div (divergence), which is $\nabla \cdot$

When it operates on a vector it yields a scalar

if $\underset{\sim}{b}(x,y,z)$ is a vector field, then

$$g(x,y,z) = \nabla \cdot \underset{\sim}{b} = \partial_x b_1 + \partial_y b_2 + \partial_z b_3$$

where $\underset{\sim}{b} = \langle b_1(x,y,z), b_2(x,y,z), b_3(x,y,z) \rangle$

The Curl (rotation) operator $\nabla \times$

When it operates on a vector, it yields a vector:

$$\nabla \times \underset{\sim}{b} = \underset{\sim}{G}(x,y,z)$$

$$\nabla \times \underset{\sim}{b} = \hat{i}(\partial_y b_3 - \partial_z b_2) - \hat{j}(\partial_x b_3 - \partial_z b_1) + \hat{k}(\partial_x b_2 - \partial_y b_1)$$

note !!

Special cases and properties

In the following, assume everything is 3D

$\underset{\sim}{r} = \langle x, y, z \rangle$

$\underset{\sim}{F} = \underset{\sim}{F}(x, y, z) = \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$

$f = f(x, y, z)$

the symbol $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_y \equiv \frac{\partial}{\partial y}$, etc...

ex) $\nabla \cdot \underset{\sim}{F} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3$, a scalar function.

ex) $\nabla \times \underset{\sim}{F} = \hat{i}(\partial_y f_3 - \partial_z f_2) - \hat{j}(\partial_x f_3 - \partial_z f_1) + \hat{k}(\partial_x f_2 - \partial_y f_1)$

, a vector.

Properties worth memorizing

$$\begin{cases} \nabla \cdot (f\underset{\sim}{F}) = f\nabla \cdot \underset{\sim}{F} + \underset{\sim}{F} \cdot \nabla f \\ \nabla \times (f\underset{\sim}{F}) = f\nabla \times \underset{\sim}{F} + \nabla f \times \underset{\sim}{F} \end{cases}$$ (remember, the order is important)

$$\begin{cases} \nabla \times \nabla f = 0 \\ \nabla \cdot (\nabla \times \underset{\sim}{F}) = \underset{\sim}{0} \end{cases}$$

More examples (3D)

ex) $\nabla \cdot \underset{\sim}{r} = \nabla \cdot \langle x, y, z \rangle = 3$

ex) if $\underset{\sim}{r} = \langle x, y, z \rangle$ and $r = \sqrt{x^2+y^2+z^2}$

$$\nabla \cdot (\underset{\sim}{r} f(r)) = 3 f(r) + r \frac{df}{dr}$$

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If $\nabla \times \underset{\sim}{V} = \underset{\sim}{0}$ we say $\underset{\sim}{V}$ is "irrotational"

If $\nabla \cdot \underset{\sim}{V} = 0$ we say $\underset{\sim}{V}$ is "incompressible"

Note: ~~Since~~ Suppose $\underset{\sim}{F} = \nabla f$. Since $\nabla \times \nabla f = 0$ we say that conservative vector fields are "irrotational".

Alternatively, suppose $\underset{\sim}{F}$ is irrotational. Then $\underset{\sim}{F} = \nabla f$ i.e. can be found as a gradient of a "potential" f.

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Suppose $\underset{\sim}{F}$ is "irrotational" & incompressible: Then

$\nabla \times \underset{\sim}{F} = \underset{\sim}{0}$ $\therefore F = \nabla f$ if $\nabla \cdot \underset{\sim}{F} = 0$ then

$$\nabla \cdot \underset{\sim}{F} = \nabla \cdot \nabla f = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle \partial_x f, \partial_y f, \partial_z f \rangle$$

$$= \partial_{xx} f + \partial_{yy} f + \partial_{zz} f = 0$$

$$\equiv \nabla^2 f \equiv \Delta f \quad \text{"Laplace's Equation"}$$

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