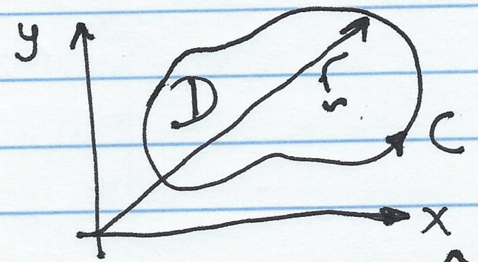
14.4 Green's Theorems: Nothing more than Stokes' Theorem in the xy plane and the Divergence Theorem in xy plane!

Suppose $\underset{\sim}{F} = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$, C is counterclockwise



$\underset{\sim}{r} = x\hat{i} + y\hat{j}$

$\nabla \times \underset{\sim}{F} = \hat{k}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$

$d\underset{\sim}{S} = \hat{k}\,dx\,dy$

So Stokes' theorem $\iint_D (\nabla \times \underset{\sim}{F}) \cdot d\underset{\sim}{S} = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

Reduces to $\boxed{\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \oint_C (f\,dx + g\,dy)}$

This is the "circulation form" of Green's Theorem because it is derived as a special case of Stokes' to find the circulation

$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

ex) Let $\underset{\sim}{F} = x\hat{j}$, C is a closed path in xy plane enclosing D, traced counter clockwise

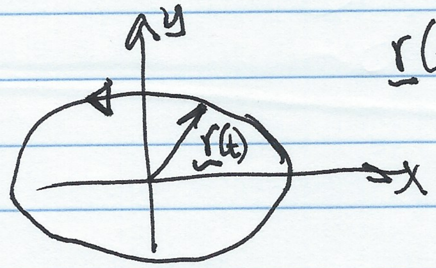
then $\oint_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \oint_C x\,dy$ & $\nabla\times \underset{\sim}{F} = \hat{k}$

$$\therefore \oint_C x\,dy = \iint_D dS = \text{Area of } D$$

For $\underset{\sim}{F} = y\hat{i}$ $\qquad \nabla\times \underset{\sim}{F} = -\hat{k}$

$$\oint_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \oint_C y\,dy = -\iint_D dS = -\text{Area of } D$$

ex) Area of ellipse : find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\underset{\sim}{r}(t) = \langle a\cos Bt, b\sin Bt\rangle$ choose $B=1$ and $0 \le t \le 2\pi$ so we go around counterclockwise & close path.

$$\text{Area of ellipse} = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$x\,dy - y\,dx = a\cos t\, b\cos t\, dt - b\sin t(-a\sin t)\,dt$$
$$= ab\,dt$$

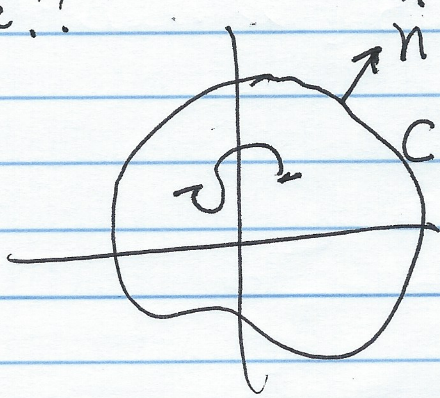
$$\therefore \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab \quad //$$

Flux Form of Green's Theorem

Comes from the divergence Theorem in 2D: the surface integral is now a line integral & the volume integral is a surface integral

$$\underbrace{\oint_C \underset{\sim}{F}\cdot\hat{n}\,ds}_{\int_S \underset{\sim}{F}\cdot d\underset{\sim}{S}} = \underbrace{\int_{\mathcal{S}} \nabla\cdot\underset{\sim}{F}\,d\mathcal{S}}_{\int_V (\nabla\cdot\underset{\sim}{F})\,dV}$$

Note: s is path and $\mathcal{S}$ is surface!!
$\mathcal{S}$ is enclosed by C
$\hat{n}$ is the outward normal to C



$$\underset{\sim}{F} = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$$

Flux form of Green's Theorem

$$\oint_C \underset{\sim}{F} \cdot \hat{n}\, ds = \iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dS$$

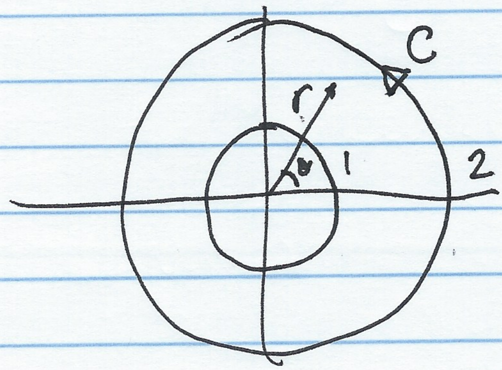
for a general curve given by $\psi(x,y) = 0$

we find $\hat{n} = \dfrac{\nabla \psi}{|\nabla \psi|}$ gives the normal.

$$\oint_C \underset{\sim}{F} \cdot \hat{n}\, ds = \oint_C f\,dy - g\,dx$$ gives the outward flux

$$\therefore \oint_C f\,dy - g\,dx = \iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dS$$

ex) Let $\underset{\sim}{F} = \langle \overset{f}{xy^2}, \overset{g}{x^2y} \rangle$ Find flux (total) across boundary of an annulus $D: \{(x,y) = 1 \le x^2+y^2 \le 4\}$



$$\nabla \cdot \underset{\sim}{F} = y^2 + x^2 = r^2$$

$$\int_S \nabla \cdot \underset{\sim}{F}\, dS = \int_0^{2\pi}\int_1^2 r\,dr\,r^2 = 2\pi \frac{r^4}{4}\Big|_1^2 = \frac{15\pi}{2}$$

or could be done by

$$\oint_C \underset{\sim}{F}\cdot\hat{n}\,ds = \oint_C f\,dy - g\,dx = \oint_C xy^2\,dy - x^2y\,dx$$

$$= \int_0^2 \sqrt{4-y^2}\,y^2\,dy + \int_2^0 -\sqrt{4-y^2}\,y^2\,dy + \int_0^{-2} -\sqrt{4-y^2}\,y^2\,dy + \int_{-2}^0 \sqrt{4-y^2}\,y^2\,dy$$

$$- \int_2^0 \sqrt{4-x^2}\,x^2\,dx + \int_0^{-2} \sqrt{4-x^2}\,x^2\,dx + \int_{-2}^0 -\sqrt{4-x^2}\,x^2\,dx + \int_0^2 -\sqrt{4-x^2}\,x^2\,dx$$

So very complicated!