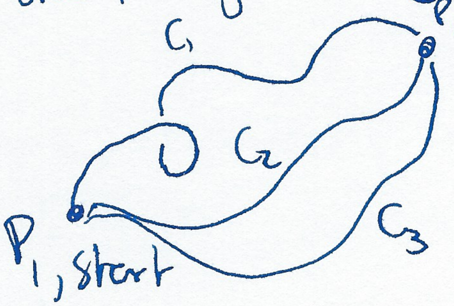
14.3 Conservative Vector Fields

For these the line integral $\int_{P_1}^{P_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = K$, a constant, that does not depend on whether C_1, C_2 or C_3 is taken.



let $\underset{\sim}{F} = P(\underset{\sim}{r})\hat{i} + Q(\underset{\sim}{r})\hat{j} + R(\underset{\sim}{r})\hat{k}$

$\underset{\sim}{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C (P\,dx + Q\,dy + R\,dz)$$

Suppose $P\,dx + Q\,dy + R\,dz = d\psi$

then $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_C d\psi = \psi(P_2) - \psi(P_1)$

(if ψ is single valued)

So what do P, Q, R need to satisfy for this to be true?

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial z}dz$$

$\therefore$ if $P = \frac{\partial \psi}{\partial x}$ $\quad Q = \frac{\partial \psi}{\partial y}$ $\quad R = \frac{\partial \psi}{\partial z}$

but $\frac{\partial P}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y}$ & $\frac{\partial Q}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial z}$ (A)

In the same way

$$\text{B}\quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \& \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \quad \text{C}$$

$\therefore$ if P, Q, R satisfy A, B, C then

$$P\,dx + Q\,dy + R\,dz = d\psi$$

In fact A, B, C are satisfied iff $\boxed{\nabla \times \underline{F} = \underline{0}}$

in that case $\displaystyle\int_C \underline{F} \cdot d\underline{r} = \int_C d\psi = \psi(\text{end}) - \psi(\text{start})$ //

For these $\displaystyle\oint_C \underline{F} \cdot d\underline{r} = 0$

$\underline{F}$ is said to be "conservative" if $\nabla \times \underline{F} = 0$.

__N.B.__ $\displaystyle\oint_C \underline{F} \cdot d\underline{r} = 0$ does not imply $\nabla \times \underline{F} = \underline{0}$.

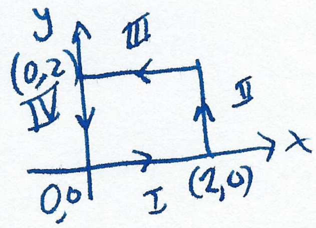
ex) let $\underset{\sim}{A} = (3x^2 + 2y^2)\hat{i} + (4xy + 6y^2)\hat{j}$

(a) compute $\nabla \times \underset{\sim}{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2+2y^2 & 4xy+6y^2 & 0 \end{vmatrix} = 0$

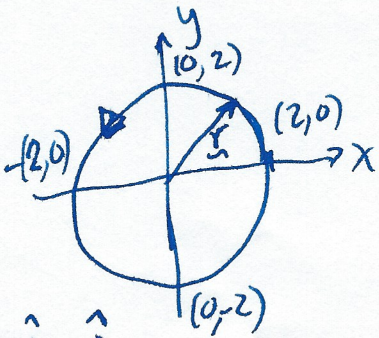
$\therefore \oint_C \underset{\sim}{A} \cdot d\underset{\sim}{r} = 0$ $\quad \underset{\sim}{A}$ is "conservative"

(b) Compute closed path integral by

(b1)



(b2)



(b1) $\oint_C \underset{\sim}{A} \cdot d\underline{r} = \int_{I} + \int_{II} + \int_{III} + \int_{IV}$ $\quad$ let $\underset{\sim}{r} = x\hat{i} + y\hat{j}$

I: $0 \le x \le 2 \quad y = 0$

$$\int_{I} \underline{A} \cdot d\underline{r} = \int_0^2 (3x^2 + \cancel{2y^2}^{\,0})\,dx + \int_0^0 (\cancel{4xy + 6y^2}^{\,0})\,dy$$

II: $x = 2 \quad 0 \le y \le 2$

$$\int_{II} = \int_2^2 (3x^2 + 2y^2)\,dx + \int_0^2 (4xy + 6y^2)\,dy = 2xy^2 + 2y^3 \Big|_{\substack{x=2 \\ y=0,2}} = 32$$

III $\quad y = 2 \quad 2 \ge x \ge 0$

$$\int_{III} = \int_2^0 (3x^2 + 2y^2)\,dx \Big|_{y=2} + \int_2^2 (4\cancel{xy} + 6y^2)^{\,0}\,dy = x^3 + 2y^2 x \Big|^0_{x=2,\,y=2} = -\cancel{24}$$

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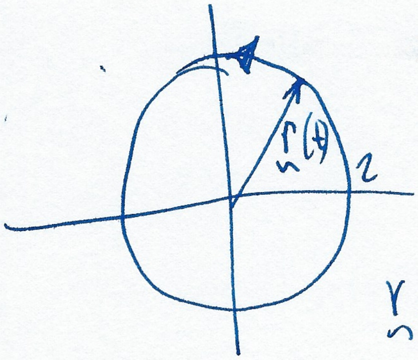
IV $\quad x = 0 \quad 2 \ge y \ge 0$

$$\int_{IV} = \int_0^0 (\cancel{3x^2 + 2y^2})^{\,0}\,dx + \int_2^0 (4\cancel{xy}^{\,0} + 6y^2)\,dy = 2y^3 \Big|_2^0 = -16$$

$\therefore \oint_C = 0$ //

(b2)



circle of radius 2
counterclockwise.

$\underset{\sim}{r} = \langle 2\cos t, 2\sin t \rangle$

$$\frac{d\underset{\sim}{r}}{dt} = 2\langle -\sin t, \cos t \rangle$$

$$\underset{\sim}{A} = (3x^2 + 2y^2)\hat{i} + (4xy + 6y^2)\hat{j}$$

$$= (6\cos^2 t + 4\sin^2 t)\hat{i} + (16\cos t\sin t + 12\sin^2 t)\hat{j}$$

$$\underset{\sim}{A} \cdot \frac{d\underset{\sim}{r}}{dt} = -2\sin t(4 + 2\cos^2 t) + 2\cos t(16\cos t\sin t + 12\sin^2 t)$$

$$\int_0^{2\pi} \underset{\sim}{A} \cdot \frac{d\underset{\sim}{r}}{dt}\, dt = 0$$ because the two terms are periodic functions of t.

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14.3

The Potential Function

Suppose $\nabla \times \underset{\sim}{F} = \underset{-}{0}$ then

$$(\ast) \quad \int_C \underset{-}{F} \cdot d\underset{\sim}{r} = \int_C d\psi = \psi(P_2) - \psi(P_1)$$

where P_2 & P_1 are ending & starting points & C is arbitrary path between these. Let $\underset{\sim}{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Also $d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz = \nabla \psi \cdot d\underset{\sim}{r}$

$\therefore$ Using $(\ast)$

$$\oint_C \underset{\sim}{F} \cdot d\underset{-}{r} - \oint_C d\psi = 0$$

or

$$\oint (\underset{\sim}{F} - \nabla \psi) \cdot d\underset{\sim}{r} = 0$$

$$\therefore \quad F - \nabla \psi \perp d\underset{\sim}{r}$$

$\therefore$ $\nabla \times \underset{\sim}{F} = 0$ implies that there exists a ψ such that $\underset{\sim}{F} = \nabla \psi$

ψ is "the Potential function"

Finding the potential function

ex) let $\underset{\sim}{F} = \langle e^x \cos y, -e^x \sin y \rangle$

check $\nabla \times \underline{F} = \underline{0}$.

then $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \oint_C d\psi = 0 \qquad \underset{\sim}{F} = \nabla\psi$

find ψ:

$\underset{\sim}{F} \cdot d\underset{\sim}{r} = e^x \cos y \, dx - e^x \sin y \, dy$

$= d\psi = \dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy$ (B)

$\therefore \dfrac{\partial \psi}{\partial x} = e^x \cos y$ (A) integrate in x $\psi = e^x \cos y + h(y)$

differentiate wrt y

$\dfrac{\partial \psi}{\partial y} = -e^x \cos y + \dfrac{dh}{dy} = -e^x \cos y$ By (B)

$\therefore \dfrac{dh}{dy} = 0 \Rightarrow h = c_0$ a constant.

$\therefore \psi = e^x \cos y + c_0$, c_0 is any constant.