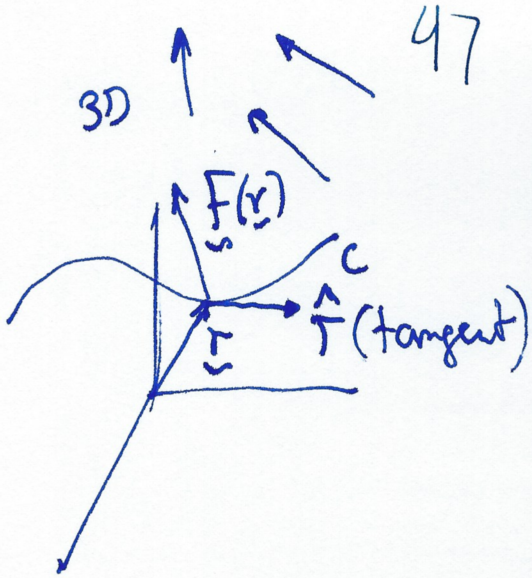
14.2 Line Integrals



The type we'll study is

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

(•) Here C is a known path, usually with a starting and ending point. The path is continuous.

(•) $\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{r})$ is a given vector field. It is defined in some region in 3D, but we only care about $\underset{\sim}{F}$ along the path

(•) here
$$\begin{cases} \underset{\sim}{r} = x\hat{i} + y\hat{j} + z\hat{k} \\ d\underset{\sim}{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} \end{cases}$$

Using Chain Rule:

$$d\underset{\sim}{r} = \frac{d\underset{\sim}{r}}{ds}\,ds = \hat{T}\,ds = \frac{d\underset{\sim}{r}}{dt}\frac{1}{\frac{ds}{dt}}\,ds$$

$$\therefore \int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_C (\underset{\sim}{F}\cdot\hat{T})\,ds$$

if $F(\underset{\sim}{r}) = P(\underset{\sim}{r})\hat{i} + Q(\underset{\sim}{r})\hat{j} + R(\underset{\sim}{r})\hat{k}$

and $\underset{\sim}{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\underset{\sim}{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_C (P\,dx + Q\,dy + R\,dz)$$

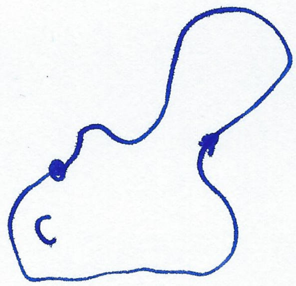
if $\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{r}(t))$

$\underset{\sim}{r} = \underset{\sim}{r}(t)$ then

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_{t_{start}}^{t_{end}} \underset{\sim}{F}\cdot\underset{\sim}{v}\,dt$$

Special Path: a closed path

$$\oint_C$$



the starting & ending points are the same.

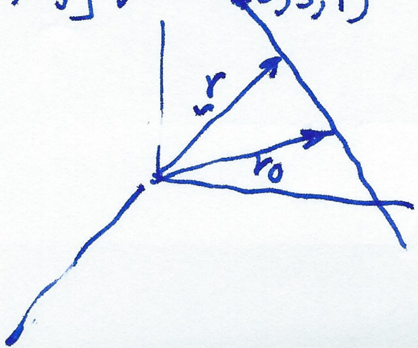
ex) if $\underset{\sim}{F}$ is a force and $\underset{\sim}{r}$ describes a path, the line integral $\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \Delta W$ is the work done by a particle subjected to a force field $\underset{\sim}{F}$ starting & ending at the ends of a path, traveling along a specified direction.

ex) Find work done by a particle moving from $(1,4,2)$ to $(0,5,1)$ in a straight line if $\underset{\sim}{F} = x\hat{i} + 3xy\hat{j} - (z+x)\hat{k}$:

(1) Parametrize curve in t:

$$\underset{\sim}{r}(t) = (1,4,2) + [(0,5,1) - (1,4,2)]t$$

$$0 \le t \le 1$$



$$\therefore\ \underset{\sim}{r}(t) = \langle \overset{x}{1-t}, \overset{y}{4+t}, \overset{z}{2-t} \rangle \quad 0 \le t \le 1$$

$$\frac{d\underset{\sim}{r}}{dt} = \langle -1, 1, -1 \rangle$$

$$F(\underset{\sim}{r}) = (1-t)\hat{i} + 3(1-t)(4+t)\hat{j} - (2-t+1-t)\hat{k}$$

$$\underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} = -3t^2 - 10t + 14$$

$$\int_{(1,4,2)}^{(0,5,1)} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 dt\, \underset{\sim}{F} \cdot \frac{d\underset{\sim}{r}}{dt} = \int_0^1 dt \left[-3t^2 - 10t + 14\right]$$

$$= -t^3 - 5t^2 + 14t \Big|_0^1 = 8 \quad //$$

Suppose we start at $(0,5,1)$ and go to $(1,4,2)$?

We can use same $\underset{\sim}{r}(t) = \langle 1-t, 4+t, 2-t \rangle$

but run t from 1 to 0!

$$\underset{\sim}{F}(\underset{\sim}{r}) \cdot \frac{d\underset{\sim}{r}}{dt} = -3t^2 - 10t + 14 \text{ as before.}$$

$$\int_1^0 dt \left[-3t^2 - 10t + 14\right] = -8$$

$\therefore$ different amount of work!

$\therefore$ Going from $(0,5,1)$ to $(1,4,2)$ or from $(1,4,2)$ to $(0,5,1)$ gives different values for ΔW. In this SPECIAL case

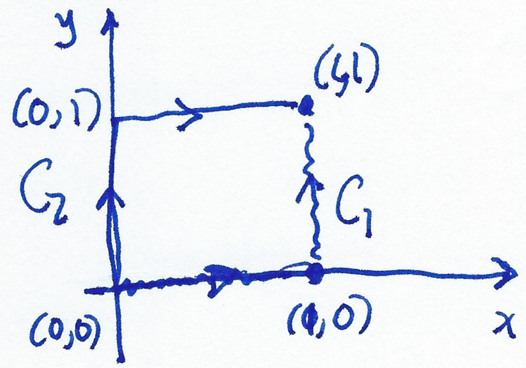
$$\Delta W\Big|_{(1,4,2)}^{(0,5,1)} + \Delta W_{(0,5,1)}^{(1,4,2)} = 0$$

. In this special case the ΔW on this close path gave 0.

More on this later when we talk about "Conservative forces".

ex) Find work done if $\underset{\sim}{F} = -\hat{i}y + \hat{j}x$

~~by a path~~ & starting point is $(0,0)$ and ending is $(1,1)$, by 2 paths



let $\underset{\sim}{r} = x\hat{i} + y\hat{j}$ $\quad d\underset{\sim}{r} = dx\,\hat{i} + dy\,\hat{j}$ $\quad \therefore W = \int_{C_i} (-y\,dx + x\,dy)$

Take Path C_1 from $(0,0)$ to $(1,0)$ to $(1,1)$:

$$W = \int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{(0,0)}^{(1,0)} \underline{F} \cdot d\underline{r} + \int_{(1,0)}^{(1,1)} \underline{F} \cdot d\underline{r}$$

$$= -\int_0^1 0\,dx + \int_0^1 1\,dy = 1$$

Take C_2 from $(0,0)$ to $(0,1)$ to $(1,1)$

$$W = \int_{C_2} \underline{F} \cdot d\underline{r} = \int_{(0,0)}^{(0,1)} \underline{F} \cdot d\underline{r} + \int_{(0,1)}^{(1,1)} \underline{F} \cdot d\underline{r}$$

$$= -\int_0^0 y\,dx + \int_0^1 0\,dy + \int_0^1 -1\,dx + \int_1^0 x\,\cancel{dy} = -1$$

$\therefore$ Work depends on path taken! //

Sometimes a line integral does not depend on the path chosen: there are certain vector fields that yield results that are path independent.