<u>Def</u>: <u>Potential function</u>: Let $\varphi(\mathbb{D})$ be a Scaler function, defined in $\mathbb{D} \subseteq \mathbb{R}^d$ d-dimensional space & be differentiable in $\mathbb{D}$. It is called a potential function if it generates a <u>gradient field</u>

$$\underset{\sim}{V} = \nabla_d \varphi(\mathbb{D}) \quad \text{a vector} //$$

ex) in 2D. Let $\varphi = \dfrac{q}{\sqrt{x^2+y^2}}$ q a charge (constant)

$r = \sqrt{x^2+y^2}$ "electric potential due to q"

The "electric field"

$$\underset{\sim}{E} = -\nabla\varphi = \langle \partial_x, \partial_y \rangle \varphi$$

$$= q\left[x(x^2+y^2)^{-3/2}\,\hat{i} + y(x^2+y^2)^{-3/2}\,\hat{j}\right]$$

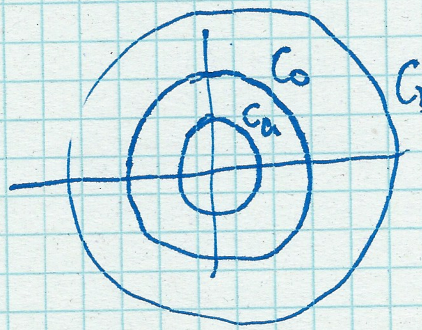
$$= q\left(\frac{x}{r^3}\,\hat{i} + \frac{y}{r^3}\,\hat{j}\right) = q\,\frac{1}{r^2}\,\hat{r}$$

EQUIPOTENTIAL CURVES & SURFACES: the level sets of a potential function are called equipotential curves.

if $\underset{\sim}{E} = \nabla\varphi \Rightarrow \underset{\sim}{E}$ is everywhere orthogonal to equipotential in 2D:

ex) let $c_0 = \sqrt{x_0^2 + y_0^2}$ be a level set:

i.e. $C_0 = \{(x,y) : x^2 + y^2 = c_0^2\}$ circles

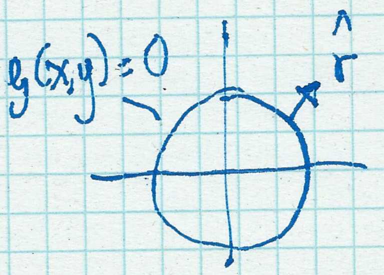


$C_a < C_0 < C_b$

levelsets

Find normal and tangential vector fields

let $g(x,y) = x^2 + y^2 - c_0^2 = 0$



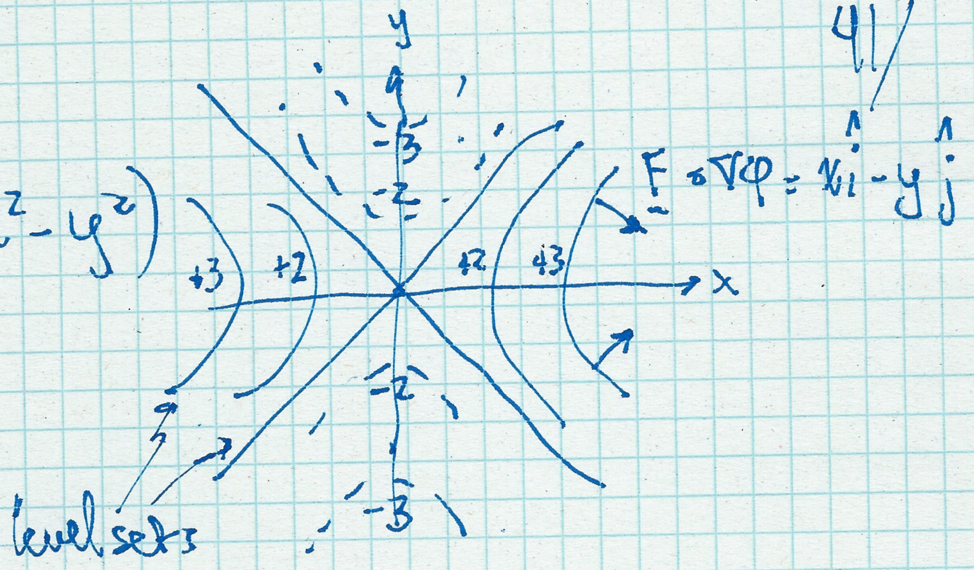
then $\nabla g = 2x\hat{i} + 2y\hat{j}$

let $\hat{n} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{2x\hat{i} + 2y\hat{j}}{\sqrt{2^2x^2 + 2^2y^2}} = \dfrac{2\,\underset{\sim}{r}}{2\,|\underset{\sim}{r}|} = \hat{r}$ (not unique)

is the normal vector. To find a tangent

$\hat{t} \cdot \hat{r} = 0 \quad \dfrac{1}{|\underset{\sim}{r}|}\left[t_1 x + t_2 y\right] = 0 \Rightarrow \begin{matrix} t_1 = -y \\ t_2 = x \end{matrix}$

$\therefore \hat{t} = \dfrac{-y\hat{i} + x\hat{j}}{|\underset{\sim}{r}|}$ (not unique) ex) $\varphi = \frac{1}{2}(x^2 - y^2)$



Slopes at any point for which $\varphi(x,y) = c$ are given by taking TOTAL DIFFERENTIAL:

$$d\varphi(x,y) = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = \frac{\partial}{\cancel{\partial}}(c) = 0$$

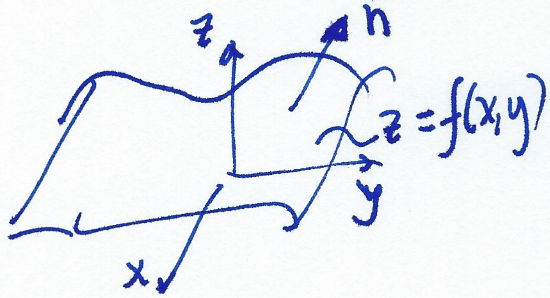
or $\nabla\varphi \cdot d\underset{\sim}{r} = 0$

$$\left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right\rangle \cdot \langle dx, dy \rangle$$

$$\frac{\partial \varphi}{\partial x}dx = -\frac{\partial \varphi}{\partial y}dy$$

$$\boxed{\therefore \frac{dy}{dx} = -\frac{\partial\varphi/\partial x}{\partial\varphi/\partial y}}$$

An important application of gradient operator: Finding the (outward/inward) unit normal vector to a surface



$$(*) \quad z = f(x,y)$$

First, turn $(*)$ into a level set equation:

$$g(x,y) = z - f(x,y) = 0$$

Since the grad yields a vector that's $\perp$ to level set then

$$\hat{n} = \frac{\nabla g}{|\nabla g|} = \frac{\hat{k} - \frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

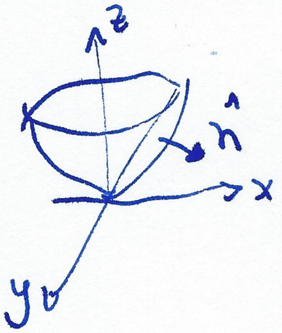
is the "outward" normal.

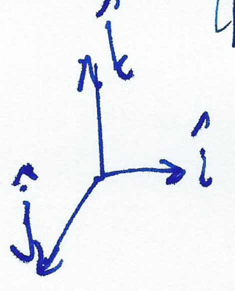
The "inward" normal: let $h(x,y,z) = -z + f(x,y) = 0$

$$\hat{n} = \frac{\nabla h}{|\nabla h|} = \frac{-\hat{k} + \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

Whether it is truly outward/inward is a matter of context.

ex)
in 3D





$z = 4x^2 + y^2 = f(x,y)$

$z \geq 0$

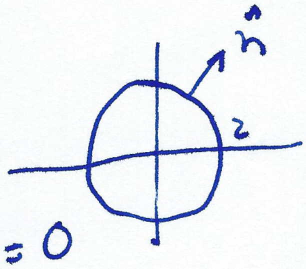
find outward normal (what we mean by "outward" is depicted in figure so we form the level set

$h(x,y,z) = 4x^2 + y^2 - z = 0$

$\hat{n} = \dfrac{\hat{i}8x + \hat{j}2y - \hat{k}}{\sqrt{1 + 8^2x^2 + 2^2y^2}}$. Note that for x, y small, $\hat{n}$ points downward....

in 2D find outward normal to a circle described by

$x^2 + y^2 = 4$



let $g(x,y) = x^2 + y^2 - 4 = 0$

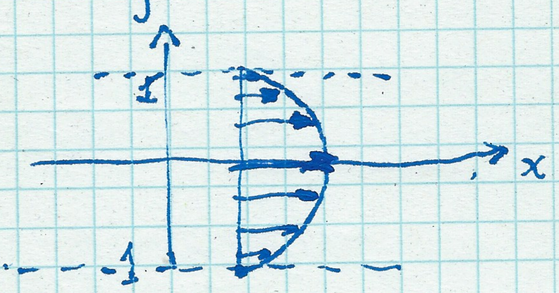
$\nabla g = 2x\hat{i} + 2y\hat{j}$ $\quad \therefore \hat{n} = \dfrac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}}$

14.1 Vector Fields

Vector field (in 3D) Let f, g & h be functions defined in $D \subseteq \mathbb{R}^3$.
A vector field in $\mathbb{R}^3$ is a vector function $\underset{\sim}{F}$ that assigns to a region D a vector $\underset{\sim}{F} = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$

ex) in 2D $\quad \underset{\sim}{v} = (1-y^2)\hat{i} + 0\,\hat{j}$

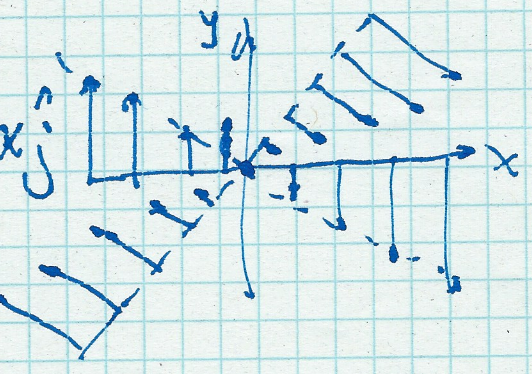
let $D = \{x \in \mathbb{R},\ y \in [-1,1]\}$



"laminar flow in a pipe"

ex) in 2D $\quad D \in \mathbb{R}^2 \qquad \underset{\sim}{v} = y\hat{i} - x\hat{j}$

A "vortex" flow

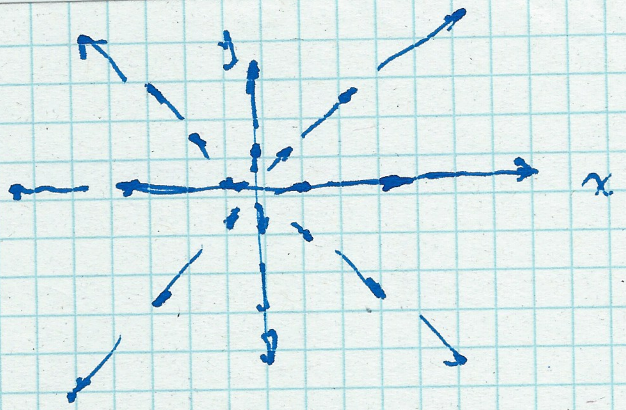


ex) The gradient of a "central field" $f = f(\sqrt{x^2+y^2}) = f(r)$

in 2D
$$\underset{\sim}{F} = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$$

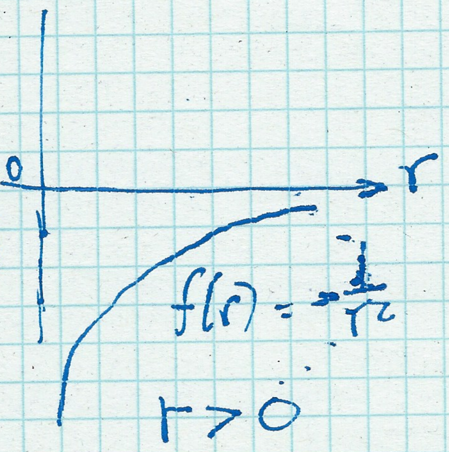
$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial r}\end{aligned}\right\} \underset{\sim}{F} = \hat{r}\frac{\partial f}{\partial r}$$

$\mathbf{F} = \hat{r}\dfrac{\partial f}{\partial r}$



a radial vector field

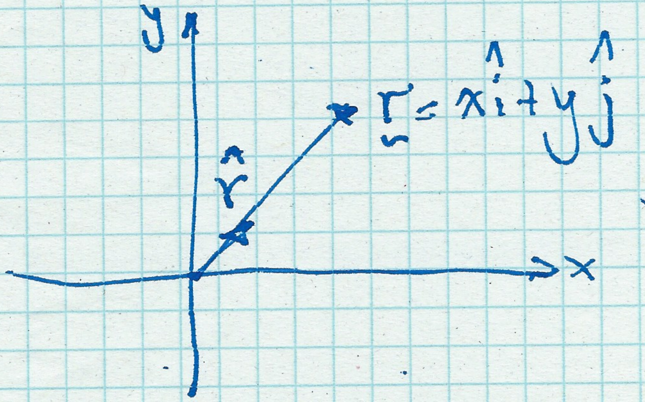
ex) let $f(r) = -\dfrac{1}{r^2}$



$r = \sqrt{x^2 + y^2}$ $\cancel{\mathbf{r} = x\hat{i} + y\hat{j}}$

if $\mathbf{F} = \nabla f(r)$

$\cancel{\mathbf{F}(\mathbf{r}) = \hat{r}\dfrac{\partial f}{\partial r}}$ $\mathbf{F}(\mathbf{r})$



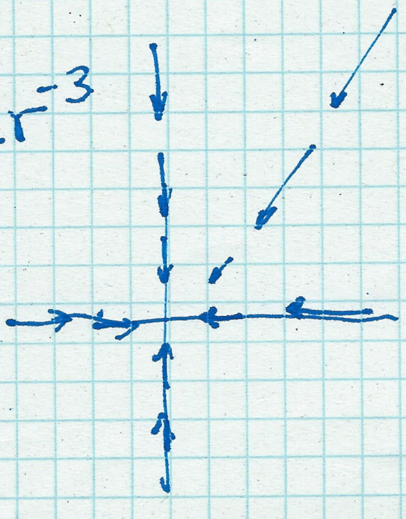
$|\mathbf{r}| = \sqrt{x^2 + y^2}$

$\hat{r} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$

if $f = r^{-2}$ $\dfrac{\partial f}{\partial r} = -2r^{-3}$

$\therefore \mathbf{F}(\mathbf{r}) = -2\hat{r}\dfrac{1}{r^3}$

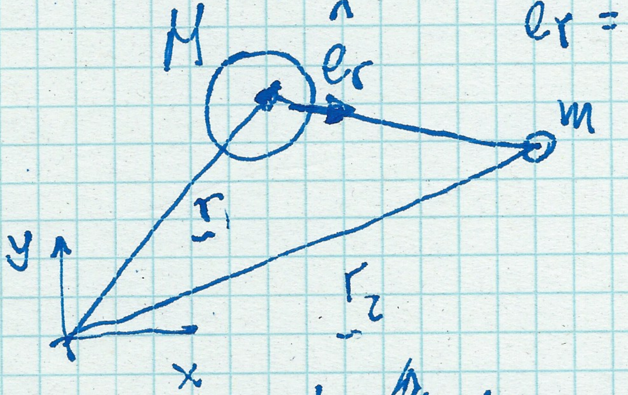
$r > 0$



~~Generally~~ Take the force of gravity between 2 masses M and m:

$$\underset{\sim}{F} = -G\frac{Mm}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|^2}\,\hat{e}_r \qquad G \text{ is a constant}$$

where $\hat{e}_r$ is a unit normal vector between $\underset{\sim}{r}_2$ and $\underset{\sim}{r}_1$



$$\hat{e}_r = \frac{\underset{\sim}{r}_2-\underset{\sim}{r}_1}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|}$$

$$\underset{\sim}{F} = -G\frac{Mm}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|^3}(\underset{\sim}{r}_2-\underset{\sim}{r}_1) = -\frac{GMm\,\hat{e}_r}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|^2}$$

"$1/r^2$ falloff of the force"

$$\underset{\sim}{F} = \nabla\varphi \quad \text{where } \varphi(r) = G\frac{Mm}{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|} = G\frac{Mm}{r}$$

is a scalar function

$$r \equiv |\underset{\sim}{r}_2-\underset{\sim}{r}_1|$$

φ is the "Gravitational potential"